Modelling and stability of supply networks

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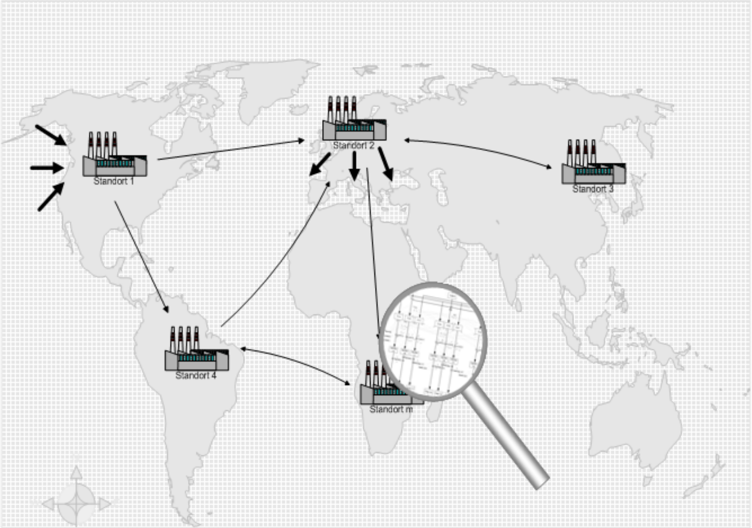
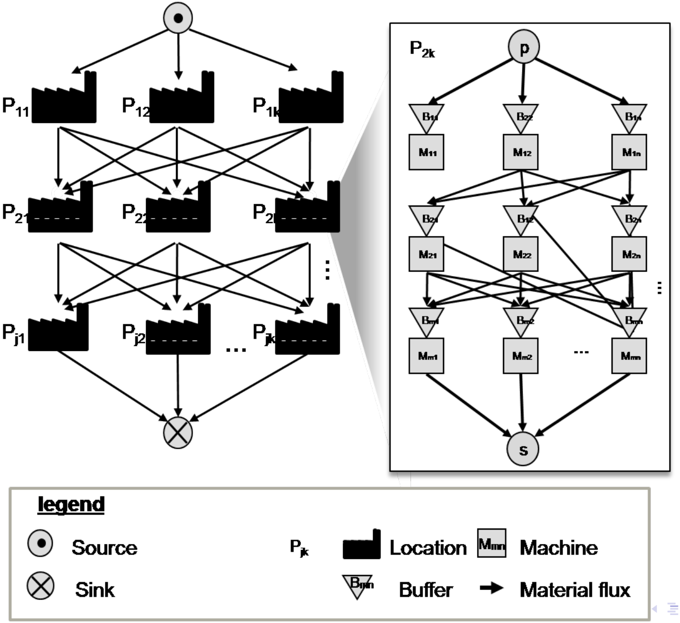
Contents

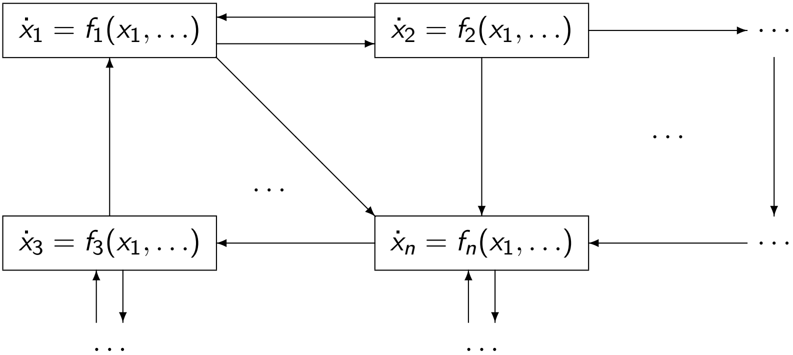
Global supply chains and logistics networks



Macro-meso-micro



Mathematical modelling



$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \dots, x_n, u_1) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \dots, x_n, u_n)
\end{aligned}
$$

Mathematical modelling

Approaches

- continuous (ODE, PDE)
- discrete
- hybrid
- time delays
- ...

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- nonlinear behavior
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$\Rightarrow$ decentralized control

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- nonlinear behavior
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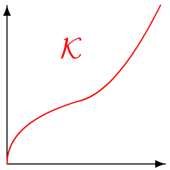
$\Rightarrow$ decentralized control

Does it work (optimality, stability, robustness, ...)?

Comparison functions

Definition

- $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called $\mathcal{K}$-function, if γ is continuous and strictly monotone increasing with $\gamma(0) = 0$
  γ is a $\mathcal{K}_\infty$-function, if it unbounded and $\gamma \in \mathcal{K}$.



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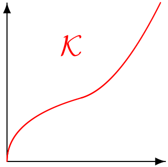
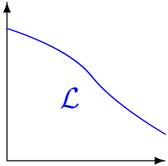
  γ is a $\mathcal{K}_\infty$-function, if it unbounded and $\gamma \in \mathcal{K}$.
- $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_+ \to \mathbb{R}_+$ is called $\mathcal{KL}$-function, if
  - β is continuous
  - $\beta(\cdot, t) \in \mathcal{K} \quad \forall t \geq 0$ and
  - $\beta(s, t) \downarrow 0$ for $t \to \infty$ and all $s \geq 0$.





Input-to-State Stability (ISS)

Definition (Sontag, 1989)

The system

$$\dot{x}(t) = f(x(t), u(t))$$

is called ISS, if there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$||x(t)|| \leq \beta(||x(0)||, t) + \gamma(||u||_\infty),$$

for all $x(0) \in \mathbb{R}^n, t \geq 0, u \in L_\infty$.

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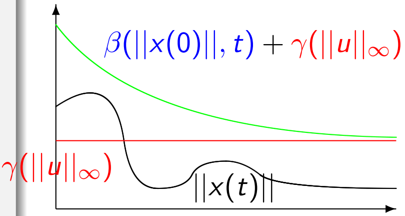
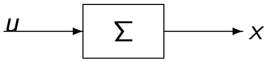


Intuitive example

$$|x(t)| < \beta(|x(0)|, t) + \gamma(||u||_\infty)$$



$\Sigma : \quad \dot{x} = f(x, u)$

u : input

x : state

input-to-state stability $\sim$

the level of x is proportional

to the level of u

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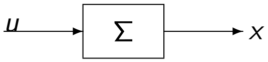

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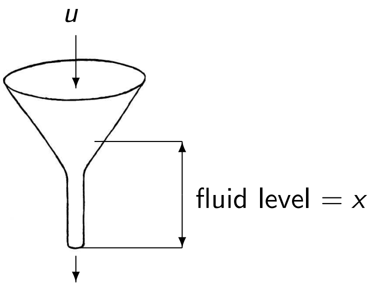

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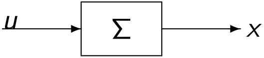

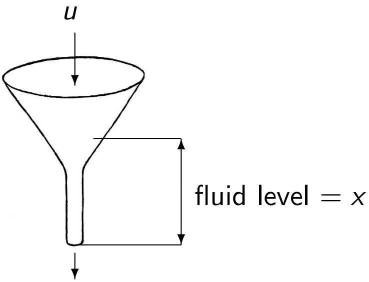

$\Sigma : \quad \dot{x} = f(x, u)$

u : input

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input-to-state stability $\sim$
the level of x is proportional
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stability $\Leftarrow$ diameter $d = g(x)$

ISS-Lyapunov function

Definition

$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function

if there are $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and a pos. def. function α such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad x \in \mathbb{R}^n,$$

$$V(x) \geq \chi(|u|) \Rightarrow \nabla V(x) f(x,u) \leq -\alpha(V(x)).$$

The function χ in then called Lyapunov-gain.

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Theorem (Sontag & Wang (1995))

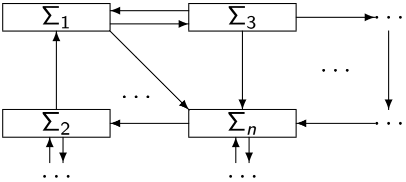
The system $\dot{x}(t) = f(x(t), u(t))$ is ISS
$\iff$ *it has an ISS-Lyapunov function.*

Network of n systems

Consider

$$\dot{x}_1 = f_1(x_1, \dots, x_n, u)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \dots, x_n, u)$$

$$f_i : \mathbb{R}^{\sum_j N_j + N_u} \to \mathbb{R}^{N_i}$$



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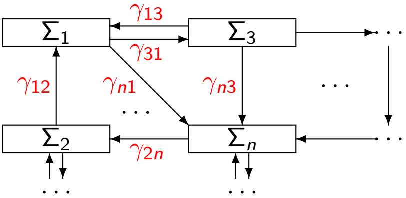

such that

$$||x_i(t)|| \leq \beta_i(||x_i(0)||, t) + \sum_{j=1}^{n} \gamma_{ij}(||x_{j\,[0,t]}||_\infty) + \eta(||u_{[0,t]}||_\infty)$$

where $\gamma_{ij} \equiv 0$ or $\gamma_{ij} \in \mathcal{K}$, and $\gamma_{ii} := 0$.

The gain-matrix

$$\Gamma := (\gamma_{ij}) = \begin{bmatrix} 0 & \gamma_{12} & \cdots & \cdots & \gamma_{1n} \\ \gamma_{21} & 0 & \gamma_{23} & \cdots & \gamma_{2n} \\ \vdots & & & & \vdots \\ \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-2} & 0 & \gamma_{n-1,n} \\ \gamma_{n1} & \cdots & \cdots & \gamma_{n,n-1} & 0 \end{bmatrix}$$

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$$\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \qquad \Gamma(s) = \begin{bmatrix} \sum_{j=1}^n \gamma_{1j}(s_j) \\ \vdots \\ \sum_{j=1}^n \gamma_{nj}(s_j) \end{bmatrix}$$

Small-gain condition for networks

Notation: $x = (x_1^T, \ldots, x_n^T)^T$ and $f = (f_1^T, \ldots, f_n^T)^T$,

for $\alpha_i \in \mathcal{K}_\infty$ let

$$D = \begin{bmatrix} (\mathrm{Id} + \alpha_1) & & \\ & \ddots & \\ & & (\mathrm{Id} + \alpha_n) \end{bmatrix}. \qquad (*)$$

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Theorem (D., Rüffer, Wirth' 2007)

If there exists D as in (*) such that

$$\Gamma \circ D(s) \not\geq s, \qquad \forall\, s \in \mathbb{R}^n_+,\ s \neq 0,$$

then the system $\dot{x} = f(x, u)$ is ISS from u to x.

Small-gain condition $\Gamma \circ D(s) \not\geq s$

- $n = 2$;
  The condition $\exists D$ with $\Gamma \circ D(s) \not\geq s \ \forall s \neq 0$ is equivalent to $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with $(\mathrm{Id} + \alpha_1) \circ \gamma_1 \circ (\mathrm{Id} + \alpha_2) \circ \gamma_2 \leq \mathrm{Id}$.
  (Jiang–Teel–Praly-condition)

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- In case γ_{ij} are linear, i.e., Γ is a nonnegative Matrix, then the condition $\exists D$ with $\Gamma \circ D(s) \not\geq s \; \forall s \neq 0$ is equivalentto $r(\Gamma) < 1$, where r is spectral radius.

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Corollary

Let Γ be linear. If

$$r(\Gamma) < 1,$$

then the interconnected system $\dot{x} = f(x, u)$ is input-to-state stable.

See also [Bailey 1966], [Rouche, Habets, Laloy 1977], [Hinrichsen, Karow, Pritchard 2005]

Associate discrete time system

Let Γ be a nonlinear operator as above. Consider

$$s_{k+1} := \Gamma(s_k), \qquad s_0 \in \mathbb{R}^n_{\geq 0}, \quad k = 0, 1, 2, \dots, \qquad (**)$$

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Theorem (D., Rüffer, Wirth' 2007)

If $\exists D$ *with* $\Gamma \circ D(s) \not\geq s$ *for all* $s \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$

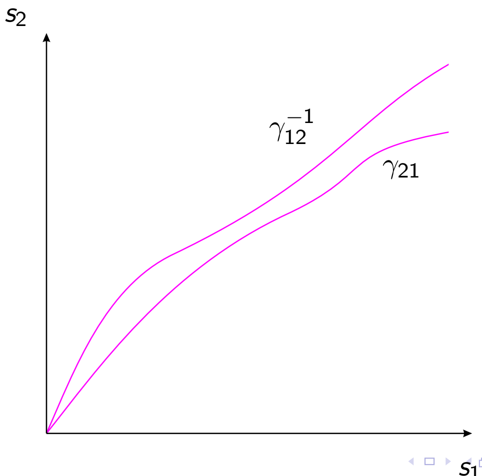
then the system (**) is globally asymptotically stabile at 0.

Geometric interpretation for $n = 2$

$$\Gamma := \begin{bmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix} \not\geq \mathrm{Id} \quad \Longleftrightarrow \quad \text{either } \gamma_{12}(s_2) < s_1 \text{ or } \gamma_{21}(s_1) < s_2$$

$$\gamma_{12} \circ \gamma_{21} < \mathrm{id}$$

Ω_1 Ω_2



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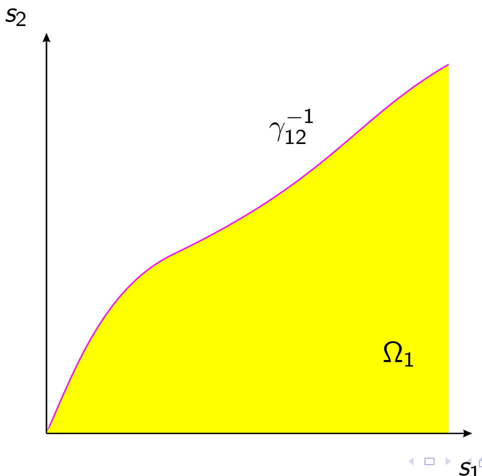

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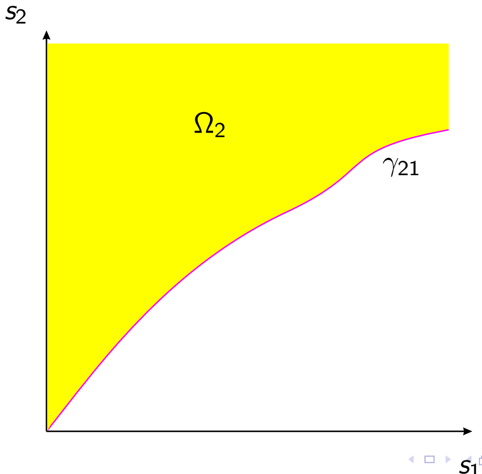

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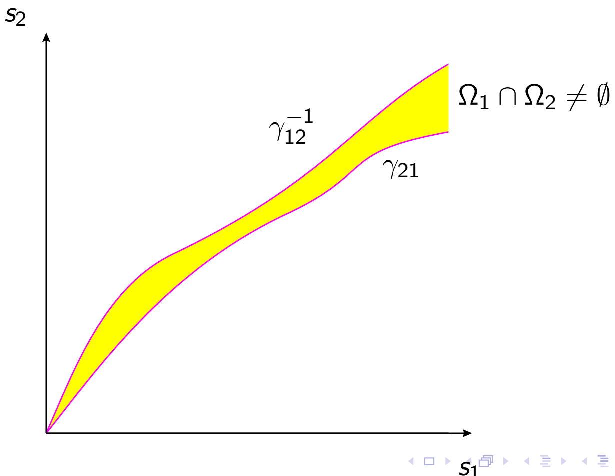

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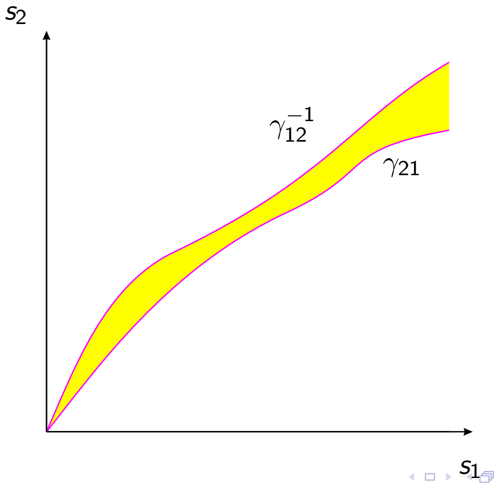

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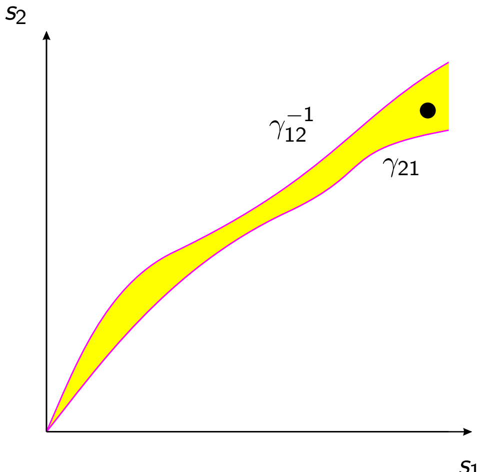

w^*

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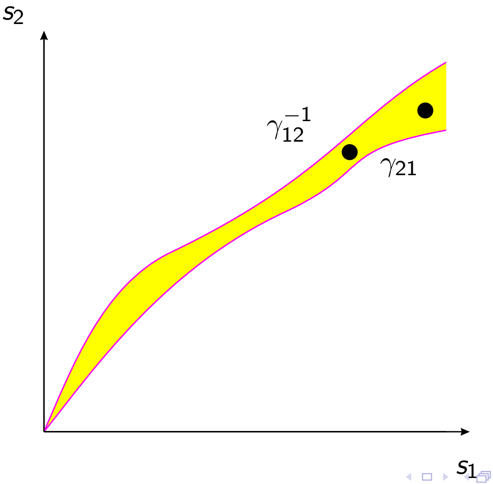

$\Gamma(w^*)$

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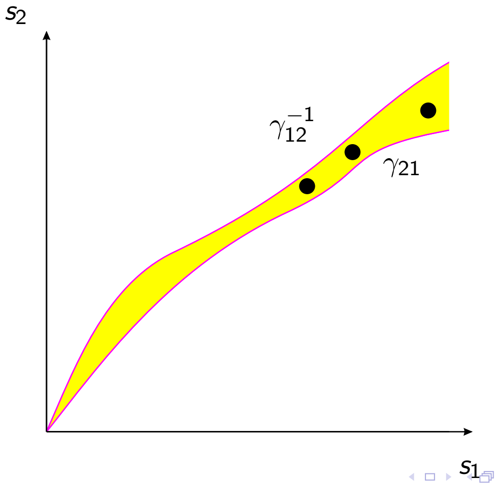

$\Gamma^2(w^*)$

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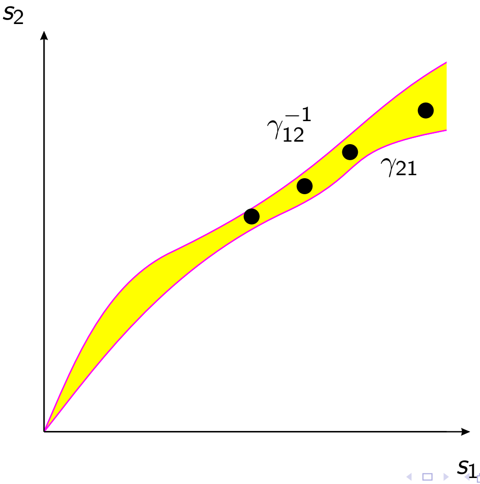

$\Gamma^3(w^*)$

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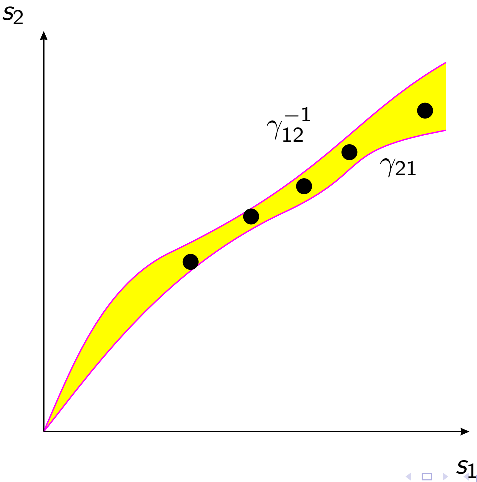

$\Gamma^4(w^*)$

Geometric interpretation for $n = 2$

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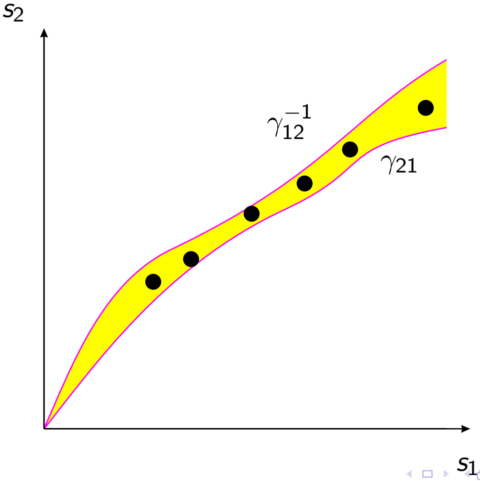

$\Gamma^5(w^*)$

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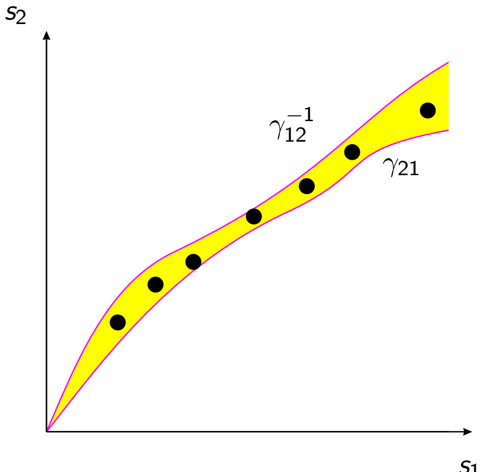

$\Gamma^6(w^*)$

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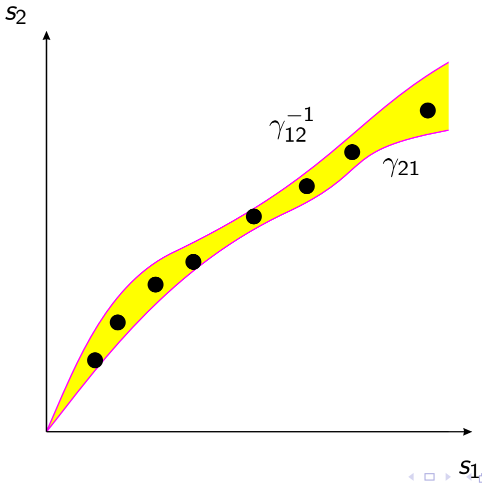
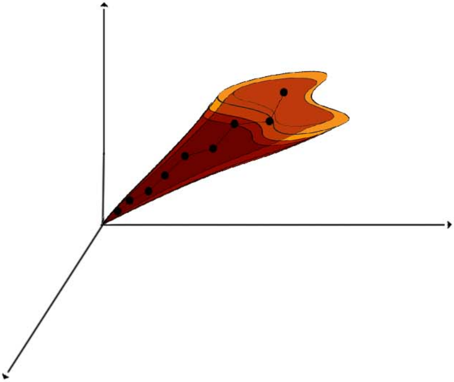


$\Gamma^7(w^*)$

Geometric interpretation



Geometric interpretation

$$\Omega_i = \left\{ x \in \mathbb{R}^N : \; |x_i| > \sum_{j \neq i} \gamma_{ij}(|x_j|) \right\}.$$

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Theorem (D., Rüffer, Wirth' 2007)

$\Gamma(s) \not\geq s \; \forall s \neq 0, s \geq 0 \Rightarrow$

- $\bigcup_{i=1}^{n} \Omega_i = \mathbb{R}^N \setminus \{0\}$
- $\Omega := \bigcap_{i=1}^{n} \Omega_i \neq \emptyset$
- Ω *connected and unbounded*
- $\Gamma(\Omega) \subset \Omega$

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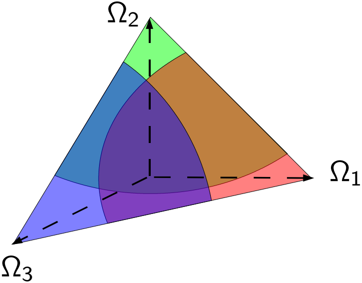

Theorem (D., Rüffer, Wirth' 2007)

$\Gamma(s) \not\geq s \; \forall s \neq 0, s \geq 0 \Rightarrow$

- $\bigcup_{i=1}^{n} \Omega_i = \mathbb{R}^N \setminus \{0\}$
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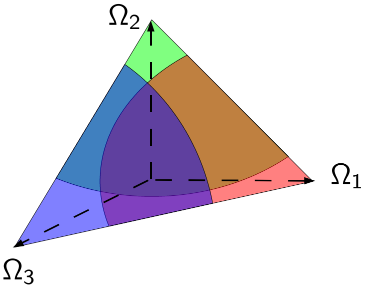

In Ω_i exists an ISS-Lyapunov function V_i with $\dot{V}_i(x) = \nabla V_i(x) \cdot f_i(x) < 0$ for $x \in \Omega_i$.

Important consequence from the small gain condition

Theorem (D., Rüffer, Wirth' 2010)

Let Γ *be irreducible. Let* $D = diag(Id + K_\infty)$ *be such that*

$$\Gamma \circ D(s) \not\geq s, \qquad \forall\, s \in \mathbb{R}^n_+,\; s \neq 0.$$

Then there exist $\mathcal{K}_\infty$*-functions* $\sigma_1, \ldots, \sigma_n$ *with* $\sigma_i' > 0,\; i = 1, \ldots, n$ *and*

$$\Gamma \circ D(\sigma(t)) < \sigma(t), \quad \forall t > 0, \quad \sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))^T$$

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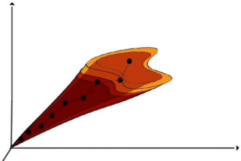

ISS-Lyapunov function for a network

Theorem (D., Rüffer, Wirth' 2010)

Let V_i be an ISS-Lyapunov function for the i-th system

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad x_i \in \mathbb{R}^{N_i},$$

$$V_i(x_i) \geq \sum_{j=1}^{n} \chi_{ij}(V_j(x_j)) + \gamma_i(|u_i|) \Rightarrow \nabla V_i(x) f_i(x, u_i) \leq -\alpha_i(V_i(x_i)),$$

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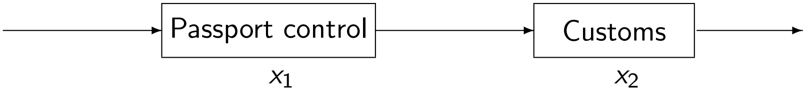
then $V(x) = \max_i\{\sigma_i^{-1}(V_i(x_i))\}$ is an ISS-Lyapunov function for the whole System $\dot{x} = f(x, u)$.

An example with two nodes



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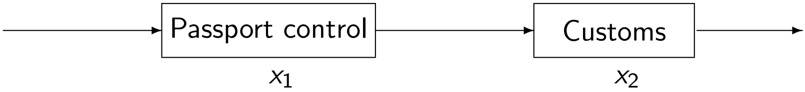

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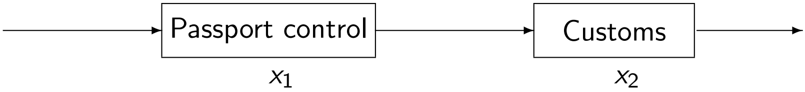

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$$b_1 = \frac{ax_1 + b\sqrt{x_1}}{1 + x_2}$$
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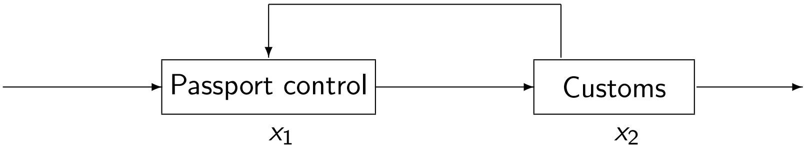

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2}$$
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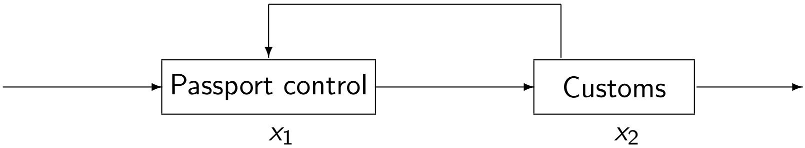

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$$\gamma_{12}(x_2) = \frac{a^2}{b^2}x_2^2$$
$$\gamma_{21}(x_1) = \left(\min\left\{\frac{c}{2a}, \frac{c^2}{b^2}\right\}\right)^{\frac{1}{2}}\sqrt{x_1}$$

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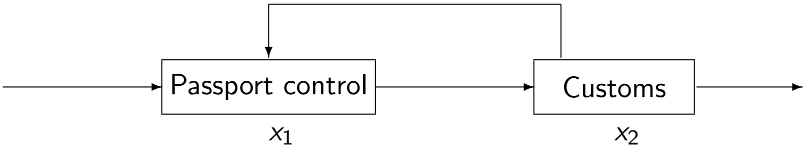

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$$\gamma_{12} \circ \gamma_{21}(s) = \frac{a^2}{b^2}\min\left\{\frac{c}{2a}, \frac{c^2}{b^2}\right\}s \leq \frac{a^2c^2}{b^4}s < s$$
$$\text{if} \quad \frac{ac}{b^2} < 1.$$

Simulation

$$u(t) = 1.5 + \sin(t) + \sin(5t)/2$$
$$x_1(0) = 10, \quad x_2(0) = 10.$$

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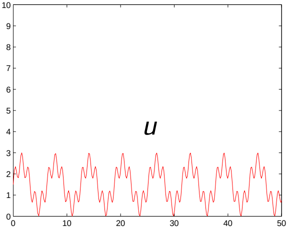


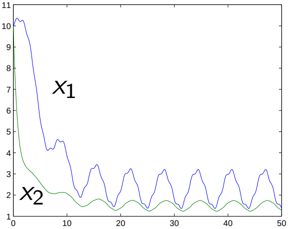
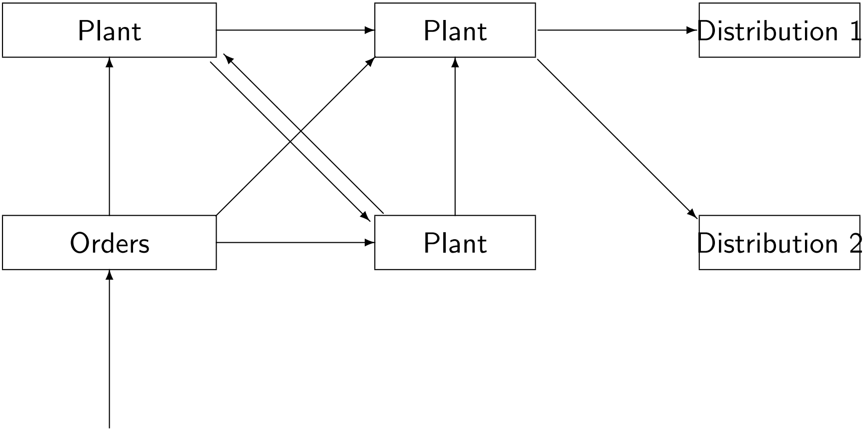


Figure: Input u and the queues x_1, x_2

Car production network



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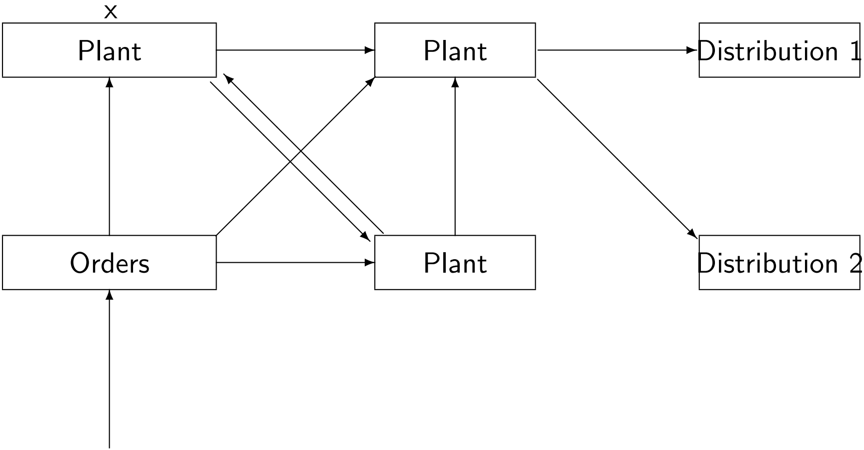

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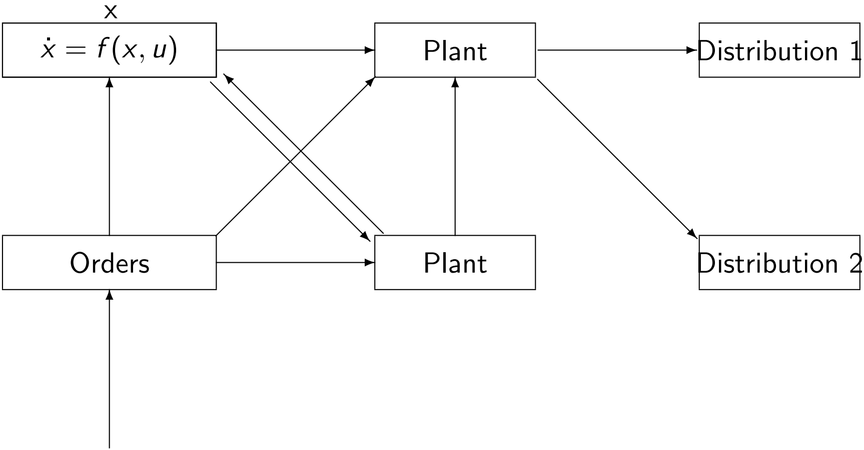

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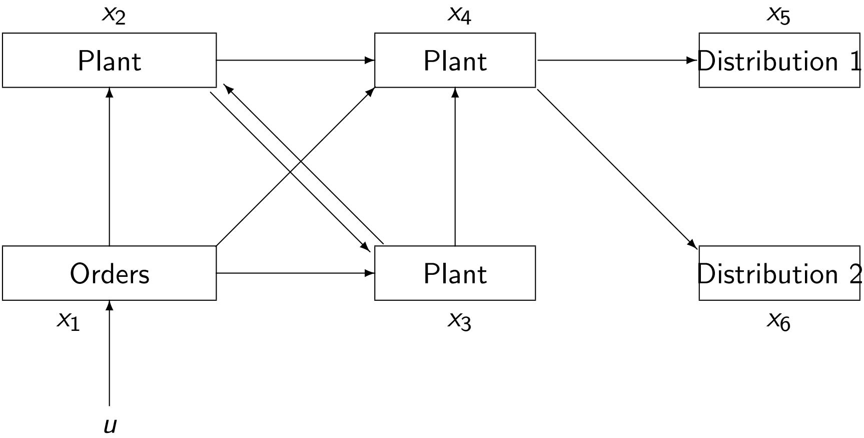

State equations

$$\dot{x}_1 = u - \frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3}$$

$$\dot{x}_2 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_3, c_3x_3\} - \min\{b_2, c_2x_2\}$$

$$\dot{x}_3 = \frac{1}{3}\frac{ax_1 + b\sqrt{x_1}}{1 + x_2 + x_3} + \frac{1}{2}\min\{b_2, c_2x_2\} - \min\{b_3, c_3x_3\}$$

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$$\dot{x}_5 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_5x_5$$

$$\dot{x}_6 = \frac{1}{2}\min\{b_4, c_4x_4\} - c_6x_6$$

Gain-Matrix

$$\Gamma := (\gamma_{ij}) = \begin{bmatrix} 0 & a_{12}x^2 & a_{13}x^2 & 0 & 0 & 0 \\ a_{21}\sqrt{x} & 0 & a_{23}x & 0 & 0 & 0 \\ a_{31}\sqrt{x} & a_{32}x & 0 & 0 & 0 & 0 \\ a_{41}\sqrt{x} & a_{42}x & a_{43}x & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54}x & 0 & 0 \\ 0 & 0 & 0 & a_{64}x & 0 & 0 \end{bmatrix}$$

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small-gain condition:

$$\Gamma(s) = \begin{bmatrix} a_{12}s_2^2 + a_{13}s_3^2 \\ a_{21}\sqrt{s_1} + a_{23}s_3 \\ a_{31}\sqrt{s_1} + a_{32}s_2 \\ a_{41}\sqrt{s_1} + a_{42}s_2 + a_{43}s_3 \\ a_{54}s_4 \\ a_{64}s_4 \end{bmatrix} \not\geq \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix}$$

Conclusions

- Logistics networks can be modelled by dynamical systems with inputs
- Stability is a fundamental property for such networks
- A stability condition for such networks is

$$\exists D : \Gamma \circ D(s) \not\geq s \quad \forall s \geq 0, s \neq 0$$

  and can be used
  - to verify stability
  - for stabilization (in particular in MPC)
- Under the small gain condition there is a method of construction of an ISS-Lyapunov function for networks
- These results are available for other types of systems

Acknowledgments

CRC 637 Autonomous Logistics