# Optimal Control for Isolated Signalized Intersections

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# Research goal

Determine sequence of signal timing plans that minimize total delay for different traffic conditions at isolated signalized intersections (ISI):

- green durations,
- switching point.

**Total delay**

$$J = \int_{t_0}^{t_f} \big(q_1(t) + q_2(t)\big)\mathrm{d}t \rightarrow \min$$

where:
$q_i(t)$: queue length for movement $i$ at time $t$

# Research issues

| | Transient control | Steady-state control |
|---|---|---|
| **Type of models** | | |
| continuous-time model | ✓ | – |
| discrete models: | | |
| discrete-event max-plus model | ✓ | ✓ |
| discrete-event piece-wise affine model | ✓ | ✓ |
| **Constraints** | | |
| lost time | ✓ | ✓ |
| maximum and minimum green durations | ✓ | ✓ |
| maximum queue length | ✓ | ✓ |
| **Optimal solution** | | |
| linear programming | – | ✓ |
| quadratic programming | ✓ | ✓ |
| mixed integer programming | ✓ | ✓ |
| Pontryagin's maximum principle | ✓ | – |
| new algorithm for solving control problem | ✓ | – |
| **Necessary and sufficient conditions** | ✓ | ✓ |

✓ = main topic of this presentation

# Transient control: problem definition

Given arrival and departure rates, and initial queue lengths, calculate control sequence that optimize a given criterion J.

### models

- continuous-time models
- discrete-event models
- deterministic and stochastic models

### traffic conditions

- undersaturated
- oversaturated

### optimization criteria

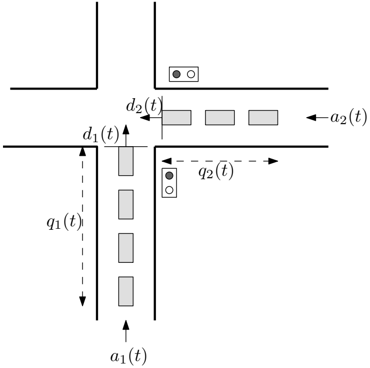
- minimum total delay
- maximum throughput

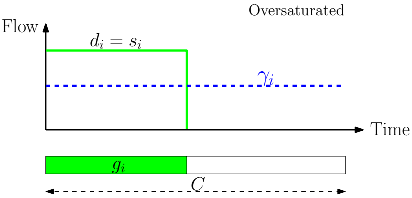
### mathematical programming problems

- classic optimization
- linear and quadratic programming
- mixed integer programming
- Pontryagin's maximum principle

# Traffic terminology

- *Green duration*, $g_i$ [s]
- *Lost time*, $L_i$ [s]
- *Cycle length*, $C$ [s]: $C = \sum_i (g_i + L_i)$
- *Queue length*, $q_i(t)$ [veh]
- *Arrival rate*, $a_i(t)$ [veh/s]
- *Saturation flow*, $s_i(t)$ [veh/s]
- *Departure rate*, $d_i(t)$ [veh/s]
- *Throughput*, $\gamma_i(t)$ [veh/s]: $\gamma_i = s_i \cdot \frac{g_i}{C}$
- *Green split*, $u_i$ [−]: $u_i = \frac{g_i}{C}$

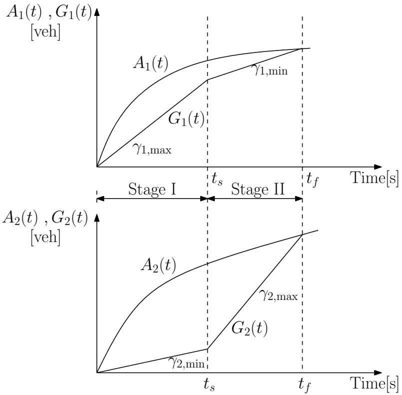


# "Bang-bang" control concept

## Assumption (Gazis and Potts, 1963; Gazis, 1964)

The total delay is minimized if the queues of all movements are dissolved simultaneously.




- two movements, $d_1 > d_2$
- *Cumulative arrival*, $A_i(t)$ [veh]: $A_i(t) = \int_0^t a_i(\tau)\mathrm{d}\tau$
- *Cumulative throughput*, $G_i(t)$ [veh]: $G_i(t) = \int_0^t \gamma_i(\tau)\mathrm{d}\tau$
- $q_i(t) = A_i(t) - G_i(t)$

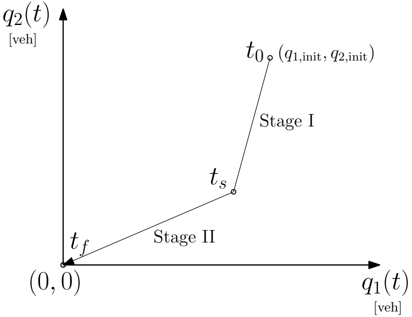
## Optimal policy

- Stage I: $\gamma_{1,\max}$ and $\gamma_{2,\min}$.
- Stage II: $\gamma_{1,\min}$ and $\gamma_{2,\max}$.

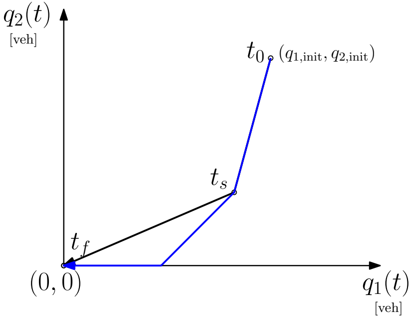
where: $t_s$ [s] switching point, $t_f$ [s] final time.

# Traffic "bang-bang" control literature survey

- Gazis and Potts (1963): optimal $t_s$ and $t_f$ are found graphically by trial and error.
- Gazis (1964): optimal "bang-bang" solution by using Pontryagin's Maximum Principle.
- Michalopoulos and Stephanopoulos (1977,78): maximum queue lengths constraint and system of two intersections.
- Chang (2000): discrete minimal delay model.



# Continuous-time models




- control variables: green splits,
- final queue lengths: $\forall i : q_i(t_f) = 0$,
- queues dissolve at the same time.

## Contributions to continuous-time models

- simultaneous dissipation assumption is relaxed:
  - new mathematical model for queue dynamics (slack variables)
  - optimal policy by PMP for new model
- deriving optimal solutions for cases with additional constraints
- state (queue length) feedback control

# Brief description of Pontryagin's Maximum Principle

**Classical optimal control problem (OCP)**

$$\int_0^T f_0(x,u)\mathrm{d}t \rightarrow \min \quad (1)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x,u) \quad (2)$$

$$x(0) = x_0,\ x(T) = x_T \quad (3)$$

$$u_{\min} \leq u(t) \leq u_{\max} \quad (4)$$

where:
*control variables* $u(t) \in \mathbf{R}^m$, *state variables* $x(t) \in \mathbf{R}^n$, $f(x,u) \in \mathbf{R}^n$, and $m \leq n$.

According to PMP:

$$H = p^T \cdot f(x,u) - f_0(x,u) \quad (5)$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p + \frac{\partial f_0}{\partial x}^T \quad (6)$$

*Hamiltonian* $= H$,
*costate variables* $p(t) \in \mathbf{R}^n$.
If $\exists (x^*, u^*) \rightarrow \exists\ p^*$ such that:

(a) $H(x^*, u^*, p^*) \geq H(x^*, u, p^*) \Rightarrow \partial H/\partial u = 0$ if $H$ is differentiable w.r.t $u$,

(b) $x^*$ and $p^*$ satisfy (2) and (6),

(c) $u^*$ satisfies (4),

(d) the end conditions in (3) must hold.

## Optimal traffic control problem (continuous-time model)

$$J = \int_{t_0}^{t_f} \big(q_1(t) + q_2(t)\big)\mathrm{d}t \rightarrow \min$$

$$\frac{\mathrm{d}q_1(t)}{\mathrm{d}t} = a_1(t) - d_1(t)\cdot u(t) + v_1(t)$$

$$\frac{\mathrm{d}q_2(t)}{\mathrm{d}t} = a_2(t) - d_2(t)\cdot \big(1 - u(t)\big) + v_2(t)$$

$$q_1(t_f) = 0,\ q_2(t_f) = 0$$

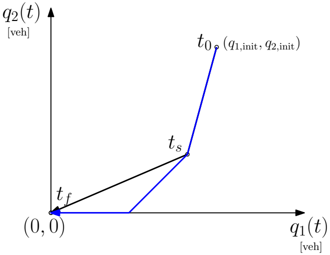
$$0 \leq q_1(t),\ 0 \leq q_2(t)$$

$$u_{\min} \leq u(t) \leq u_{\max}$$

where:
control variable $u(t)$, state variables $q_1(t)$, $q_2(t)$, and artificial slack variables $v_1(t) \geq 0,\ v_2(t) \geq 0$.

## Assumption

$\forall t,\ a_i(t) = a_i$ and $d_i(t) = d_i,\ i = 1, 2.$




from condition of non-increasing queue lengths:

$$0 \leq v_1(t) \leq v_{1,\max}(t)$$

$$0 \leq v_2(t) \leq v_{2,\max}(t)$$

where

$$v_{1,\max}(t) = \max\left[0, d_1\cdot u(t) - a_1\right]$$

$$v_{2,\max}(t) = \max\left[0, d_2 - d_2\cdot u(t) - a_2\right]$$

# Optimal control solution for the continuous-time model

- The *Hamiltonian* function, $H$, is formed as

$$H = p_1(t) \cdot a_1 + p_2(t) \cdot a_2 + \big(p_2(t) \cdot d_2 - p_1(t) \cdot d_1\big) \cdot u(t) +$$
$$+ p_1(t) \cdot v_1(t) + p_2(t) \cdot v_2(t) - p_2(t) \cdot d_2 - q_1(t) - q_2(t)$$

where $p_1(t),\ p_2(t)$ satisfy

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\frac{\partial H}{\partial q_1} = 1$$
$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = -\frac{\partial H}{\partial q_2} = 1$$

- The optimal control solution obtained by $\max_{u,v_1,v_2} H$

$$v_1(t) = \begin{cases} v_{1,\max}(t) & \text{if } p_1(t) > 0, \\ 0 & \text{if } p_1(t) < 0, \end{cases}$$

$$v_2(t) = \begin{cases} v_{2,\max}(t) & \text{if } p_2(t) > 0, \\ 0 & \text{if } p_2(t) < 0, \end{cases}$$

$$u(t) = \begin{cases} u_{\max} & \text{if } S(t) > 0, \\ u_{\min} & \text{if } S(t) < 0, \end{cases}$$

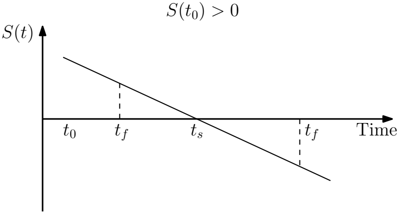
where the *switching function* $S(t) \triangleq p_2(t) \cdot d_2 - p_1(t) \cdot d_1$.

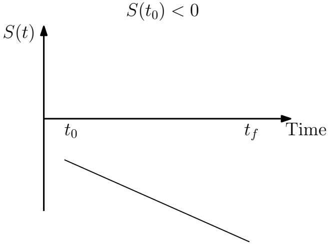
# Switching function $S(t)$

**Assumption**

$d_1 > d_2$

$$\left.\begin{aligned} S(t) &\triangleq p_2(t)\cdot d_2 - p_1(t)\cdot d_1 \\ \frac{dp_1}{dt} &= 1 \\ \frac{dp_2}{dt} &= 1 \end{aligned}\right\} \Rightarrow dS(t)/dt = d_2 - d_1 < 0$$




- at switching point $S(t_s) = 0$,
- $S(t_0) \leq 0 \rightarrow$ no switching point,
- $S(t_0) > 0$
  - initial queue lengths are such that $t_f > t_s \rightarrow$ *a single* switching point,
  - initial queue lengths are such that $t_f \leq t_s \rightarrow$ no switching point.

# Optimal control cases

$$u_{\min} \le u(t) \le u_{\max}$$

$$J = \int_{t_0}^{t_f} (q_1(t) + q_2(t))\,\mathrm{d}t \to \min$$
$$\frac{\mathrm{d}q_1(t)}{\mathrm{d}t} = a_1(t) - d_1(t)\cdot u(t) + v_1(t)$$
$$\frac{\mathrm{d}q_2(t)}{\mathrm{d}t} = a_2(t) - d_2(t)\cdot (1 - u(t)) + v_2(t)$$

## Definitions

$u_{\mathrm{L}} \triangleq \frac{a_1}{d_1}$ ; $u_{\mathrm{H}} \triangleq \frac{d_2 - a_2}{d_2}$

## Necessary condition for

- decreasing both queue lengths

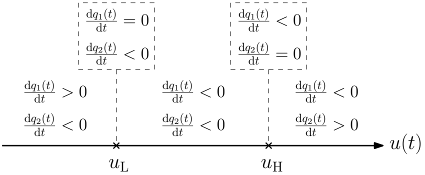
$$\frac{a_1}{d_1} + \frac{a_2}{d_2} < 1 \text{ or } u_{\mathrm{L}} < u_{\mathrm{H}}$$

I $u_{\mathrm{L}} < u_{\min} < u_{\mathrm{H}} < u_{\max}$

II $u_{\mathrm{L}} < u_{\min} < u_{\max} < u_{\mathrm{H}}$

III $u_{\min} < u_{\mathrm{L}} < u_{\max} < u_{\mathrm{H}}$

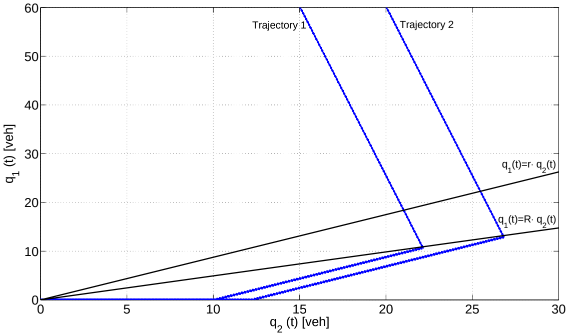
IV $u_{\min} < u_{\mathrm{L}} < u_{\mathrm{H}} < u_{\max}$




| | $u < u_{\mathrm{L}}$ | $u_{\mathrm{L}} < u < u_{\mathrm{H}}$ | $u > u_{\mathrm{H}}$ |
|---|---|---|---|
| Case $I$: | | $u_{\min}$ | $u_{\max}$ |
| Case $II$: | | $u_{\min}$ $u_{\max}$ | |
| Case $III$: | $u_{\min}$ | $u_{\max}$ | |
| Case $IV$: | $u_{\min}$ | | $u_{\max}$ |

- subcases: no switching point (function of initial queue lengths).

# Case I: $u_{\mathrm{L}} < u_{\min} < u_{\mathrm{H}} < u_{\max}$




## Switching line

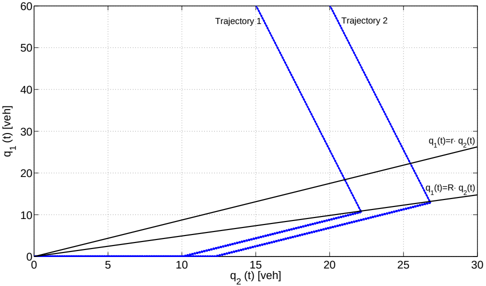
- Gazis (1964): $q_1(t) = \text{r} \cdot q_2(t)$
- Haddad (2010): $q_1(t) = \text{R} \cdot q_2(t)$

$$r = \frac{d_1 \cdot (u_{\min} - u_{\mathrm{L}})}{d_2 \cdot (u_{\mathrm{H}} - u_{\min})}$$

$$R = \frac{u_{\min} - u_{\mathrm{L}}}{u_{\mathrm{H}} - u_{\min}}$$

$$\Rightarrow \quad r = d_1/d_2 \cdot R$$

# State feedback control for Case I




$$\begin{bmatrix} u(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} u_{\max} \\ 0 \\ 0 \end{bmatrix} & \text{if} \quad q_1(t) > R \cdot q_2(t), \quad q_1(t) > 0, \quad q_2(t) > 0, \\ \begin{bmatrix} u_{\min} \\ 0 \\ 0 \end{bmatrix} & \text{if} \quad q_1(t) \leq R \cdot q_2(t), \quad q_1(t) > 0, \quad q_2(t) > 0, \\ \begin{bmatrix} u_{\min} \\ v_{1,\max}(t) \\ 0 \end{bmatrix} & \text{if} \quad q_1(t) \leq R \cdot q_2(t), \quad q_1(t) = 0, \quad q_2(t) \geq 0. \end{cases}$$

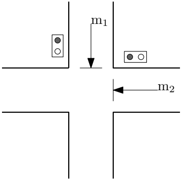
# Numerical comparison example for Case II

| **Flow rates**: | $a_1$ | $a_2$ | $d_1$ | $d_2$ |
|---|---|---|---|---|
| [veh/s] | 0.15 | 0.1 | 0.55 | 0.3 |

| **Green splits**: | $u_L$ | $u_H$ | $u_{min}$ | $u_{max}$ |
|---|---|---|---|---|
| [−] | 0.2727 | 0.6667 | 0.4 | 0.5 |

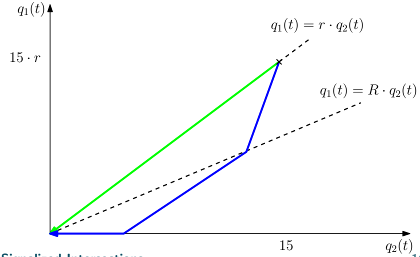
| **Initial queues**: | $q_1(t_0)$ | $q_2(t_0)$ |
|---|---|---|
| [veh] | $r \cdot 15$ | 15 |

| **Coefficient**: | $r$ | $R$ |
|---|---|---|
| [−] | 0.875 | 0.4773 |

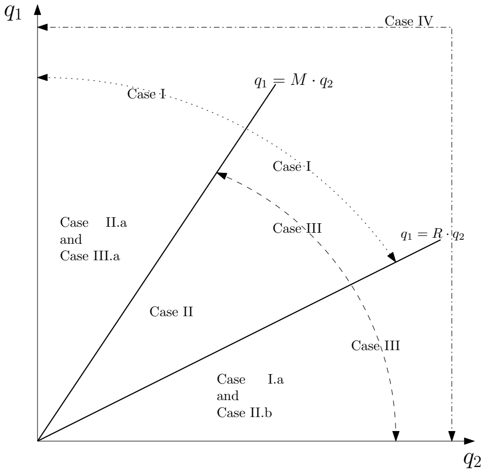



1. Trajectory that switches $u_{max}$ to $u_{min}$ at $q_1(t)/q_2(t) = r$ (Gazis 1964),
2. Optimal trajectory that switches $u_{max}$ to $u_{min}$ at $q_1(t)/q_2(t) = R$ (Haddad 2010).

| **Total delay**: | $J_1$ | $J_2$ |
|---|---|---|
| [veh · s] | 2636.7 | 2334.4 |



# Summary of optimal control cases




$$M = \frac{\mathrm{d}q_1/\mathrm{d}t}{\mathrm{d}q_2/\mathrm{d}t}\Big|_{u=u_{\max}} = \frac{a_1 - d_1 \cdot u_{\max}}{a_2 - d_2 \cdot (1 - u_{\max})}$$

function of:

- relative relation between the green split bounds,
- initial queue lengths.

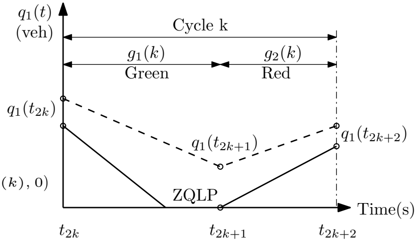
# Queue lengths at cycle $k$ for $\mathrm{m}_1$

- at the end of red light:

$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + a_1(t_{2k+1}) \cdot g_2(k)$$

- at the end of green light:

$$q_1(t_{2k+1}) = \max\left(q_1(t_{2k}) + (a_1(t_{2k}) - d_1(t_{2k})) \cdot g_1(k), 0\right)$$




- Discrete-event Max-Plus (DMP) problem

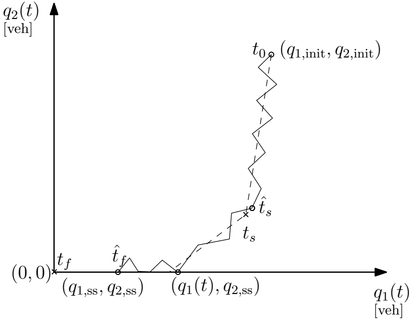
$$\min_{g_1(0),g_2(0),g_1(1),g_2(1),\cdots,g_1(N-1),g_2(N-1)} J$$

subject to

$$q_1(t_{2k+1}) = \max\left(q_1(t_{2k}) + (a_1(t_{2k}) - d_1(t_{2k})) \cdot g_1(k), 0\right)$$
$$q_1(t_{2k+2}) = q_1(t_{2k+1}) + a_1(t_{2k+1}) \cdot g_2(k)$$
$$q_2(t_{2k+1}) = q_2(t_{2k}) + a_2(t_{2k}) \cdot g_1(k)$$
$$q_2(t_{2k+2}) = \max\left(q_2(t_{2k+1}) + (a_2(t_{2k+1}) - d_2(t_{2k+1})) \cdot g_2(k), 0\right)$$

for $k = 0, 1, 2, \dots, N-1$.

# Discrete-event models




- the "saw tooth" profile evolution,
- two time instants: $\hat{t}_s$ and $\hat{t}_f$ coincide exactly with an end of a cycle.

## Contributions to discrete-event models

- two decision variables per cycle: green split and cycle length,
- relaxing simultaneous dissipation assumption,
- modeling zero-queue length periods (ZQLP's),
- final queue lengths = steady-state queues.

# Optimal discrete formulations

**modeling ZQLP**

- max-plus equations: $a = \max(b + c, 0)$ where $a, b, c \in \mathbb{R}$.
- piece-wise affine equations: $g(x) = \begin{cases} f(x) & \text{if } f(x) > 0, \\ 0 & \text{if } f(x) \leq 0. \end{cases}$

Two discrete-event problems:

1. **discrete-event max-plus problem:**
   max-plus equations $\longrightarrow$ linear inequality equations:

$$a = \max(b + c, 0) \Rightarrow \begin{cases} a \geq b + c \\ a \geq 0 \end{cases}$$

2. **discrete-event Piece-Wise Affine (PWA) problem:**
   PWA equations $\longrightarrow$ mixed integer equations
   - a nonlinear $J$: mixed-integer programming (MIP) algorithms,
   - a linear $J$: mixed-integer linear programming (MILP) algorithms.

# Contributions

- **A continuous-time model:** introduction of slack-variables,
- **Discrete-event models:** discrete-event piecewise and discrete-event max-plus models. The models are mathematically capable to describe ZQLP's,
- **Pontryagin's Maximum Principle:** the PMP is used to solve the optimal control problem for ISI,
- **LP, SQP, MIP, MILP:** different mathematical methods can solve the traffic control problems,
- **Feedback control:** the optimal control law is formulated in feedback form as a function of current queue lengths,
- **Dissipation pattern:** the simultaneous dissipation assumption on the optimal solution of the queue lengths is relaxed,
- **Switching line:** the switching line of the optimal trajectory in the bang-bang control proposed by Gazis (1964) is adjusted.

# Future research

- Model extension for isolated signalized intersections,
- Optimal control for urban traffic systems,
- Hybrid dynamical models for traffic networks.

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Conference papers:

- J. Haddad, B. De Schutter, D. Mahalel, and P. O. Gutman, "Steady-state and N-stages control for isolated controlled intersections," in *Proceedings of the 2009 American Control Conference*, St. Louis, MO, USA, June 2009, pp. 2843–2848.
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