A distributed NMPC scheme without stabilizing terminal constraints

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Setup

We consider $m \geq 1$ nonlinear discrete time control systems

$$x_k(n+1) = f(x_k(n), u_k(n)), \quad k = 1, \ldots, m$$

with $x_k(n) \in X_k$, $u_k(n) \in U_k$, $\quad X_k$, U_k metric spaces

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Before we give the precise problem formulation we illustrate the problem by a simple example

Example: simple mobile robots in the plane

position: $x_k = (x_{k,1}, x_{k,2}) \in [-1,1]^2$

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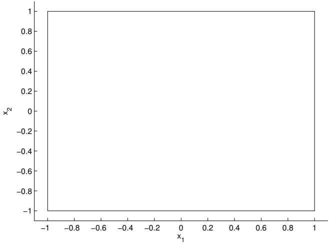

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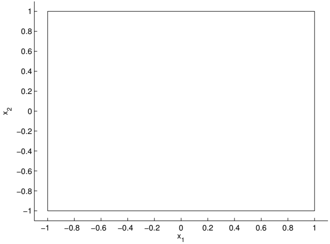
sampling time: $T > 0$

$$\begin{aligned} x_{k,1}(n+1) &= x_{k,1}(n) + T u_{k,1}(n) \\ x_{k,2}(n+1) &= x_{k,2}(n) + T u_{k,2}(n) \end{aligned}$$



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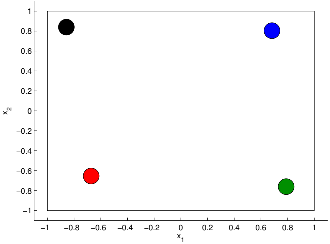

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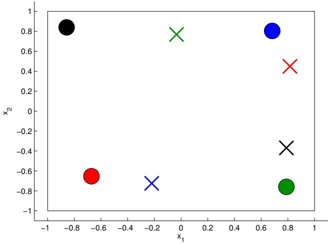

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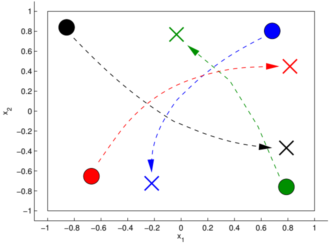

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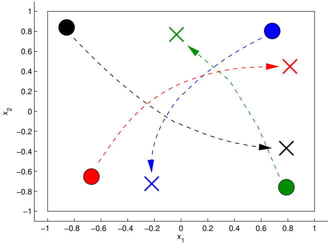

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- while staying in $[-1, 1]^2$ and avoiding collisions (state constraints)

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$$x_k(n+1) = f(x_k(n), F_k(x_k(n), y_k(n)))$$

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Idea: use model predictive control (MPC)

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How can we guarantee asymptotic stability of the closed loop?

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or variants like $x^u(N) \in \mathcal{N}(x^\star)$ plus terminal costs

(regional constraint, [Chen/Allgöwer '98, ...])

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Regional terminal constraints may allow for smaller N but the construction of terminal costs becomes difficult if the terminal region contains an obstacle

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Then F stabilizes the system under the state constraints $\mathbb{X}$ if

$$\alpha = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)} > 0, \quad \text{where } \gamma_i = \sum_{k=0}^{i-1} C\sigma^k$$

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In the one-robot problem with $\ell(x, u) = \|x - x^\star\|^2 + \|u\|^2/10$, in simulations stability is obtained for $N = 3$

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Common idea in the proofs

Both with and without stabilizing terminal constraints, the stability proof relies on establishing the inequality

$$V_N(x(n+1)) < V_N(x(n))$$

for the optimal value function $V_N(x) = \inf_{u \in \mathcal{U}} J_N(x,u)$ in a suitable uniform way

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Common idea in the proofs

Both with and without stabilizing terminal constraints, the stability proof relies on establishing the inequality

$$V_N(x(n+1)) < V_N(x(n))$$

for the optimal value function $V_N(x) = \inf_{u \in \mathcal{U}} J_N(x,u)$ in a suitable uniform way $\rightsquigarrow$ V_N is a Lyapunov function

To this end, the proofs use the tail

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$\rightsquigarrow$ crucial for extension to the distributed context: $x_k^{opt}(1), \ldots, x_k^{opt}(N)$ for subsystem x_k must remain feasible when the other subsystems x_l, $l \neq k$, update their prediction

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$$(x_1^{opt,n}(j), \ldots, x_k^{opt,n}(j), x_{k+1}^{opt,n-1}(j+1), \ldots, x_m^{opt,n-1}(j+1)) \in \mathbb{X}$$

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What is Y_k in this scheme?

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How is the scheme initialized?

At time $n = 0$ we start with arbitrary, i.e., not necessarily optimal feasible solutions. These can be found by optimization or any other method

Stability theorem with terminal constraints

Theorem [Richards/How]: Assume that for the initial values $x_k(0)$, $k = 1, \ldots, m$, we can find feasible solutions, i.e., $x_k^u(\cdot)$ with $x_k^u(0) = x_k(0)$, $x_k^u(N)$ satisfying the terminal constraints and

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- all “conflicts” are resolved in the initialization phase

Removing the terminal constraints

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Question: What is a suitable "distributed" controllability condition?

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Question: What is a suitable "distributed" controllability condition?

For defining such a condition, we need some more notation

Partial state constraints

Recall: the state space of the overall system is

$$X = X_1 \times X_2 \times \ldots \times X_m$$

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For an index set $I = \{i_1, \ldots, i_p\} \subseteq M := \{1, \ldots, m\}$ define the partial state space

$$X_I := X_{i_1} \times X_{i_2} \times \ldots \times X_{i_p}$$

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The partial state constraint set is defined as

$$\mathbb{X}_I := \{x_I \in X_I \,|\, \text{there is } x_{M\setminus I} \in X_{M\setminus I} \text{ with } (x_1, \ldots, x_m) \in \mathbb{X}\}$$

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Distributed controllability

In words: "no matter what the others intend to do, the k-th subsystem can find a feasible way towards its equilibrium, provided it knows what the others intend to do"

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Formally: at each time $n \geq 0$, denote the other subsystems' predictions available to the k-th subsystem by $x_{I_j}^{opt}(j)$, where $I_j \subset M$ with $k \notin I_j$ and the time arguments are already appropriately shifted.

Then we assume that there are $C > 0$, $\sigma \in (0,1)$ such that for each $j = 0, \ldots, N-2$ there is $u_k(\cdot)$ with

$$(x_{I_j}^{opt}(j+j'), x_k^u(j')) \in \mathbb{X}_{I_{j+j'} \cup \{k\}}$$

and

$$\ell_k(x_k^u(j'), u_k(j')) \leq C\sigma^{j'} \min_u \ell_k(x_k^u(0), u)$$

for $j' = 0, \ldots, N-j-1$ where $x_k^u(0) = x_k^{opt,n-1}(j+1)$.

Stability theorem without terminal constraints

Assume that the distributed controllability assumption holds.

Then the F_k stabilize all subsystems under the state constraints $\mathbb{X}$ if

$$\alpha = 1 - \frac{(\gamma_N - 1) \prod_{i=2}^{N} (\gamma_i - 1)}{\prod_{i=2}^{N} \gamma_i - \prod_{i=2}^{N} (\gamma_i - 1)} > 0, \quad \text{where } \gamma_i = \sum_{k=0}^{i-1} C\sigma^k$$

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In the 4 robot problem with $\ell_k(x,u) = \|x - x_k^\star\|^2 + \|u\|^2/10$, in simulations stability is obtained for $N = 5$ up to $N = 8$, depending on the initial configuration

Conclusion and discussion

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Conclusion and discussion

- The hierarchical scheme can be extended to NMPC without stabilizing terminal constraints
- This may lead to considerably shorter optimization horizons N
- Conflicts are resolved when they are detected, not necessarily in the initialization phase
- However, there are many open questions regarding both the controllability assumption and the design of the scheme. Some of these will be discussed on the remaining three slides — and maybe during and after this workshop

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Discussion: distributed controllability

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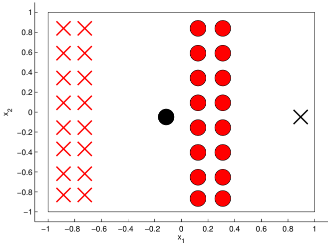
- a sufficient condition is that controllability holds for all possible trajectories — this is easier to check but very restrictive

- In the robot example, this sufficient condition holds if m is relatively small and there are no encounters at the boundary of the state space

- However, even for large m, in simulations the scheme works without problems. A possible explanation is given by the following scenario.

Discussion: distributed controllability



large m

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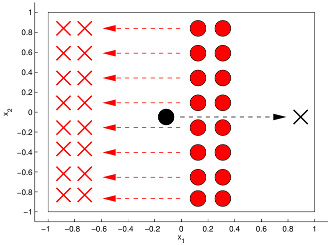

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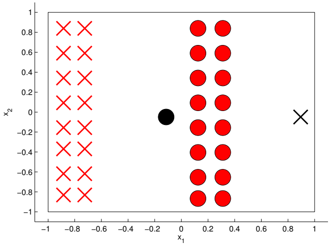

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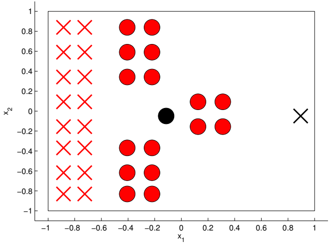

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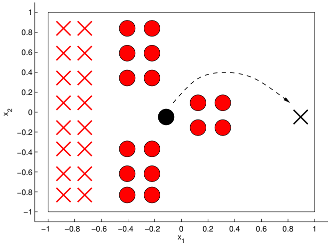

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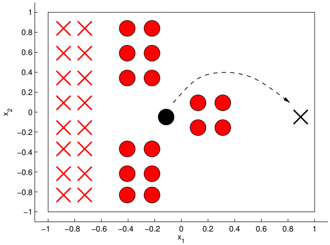

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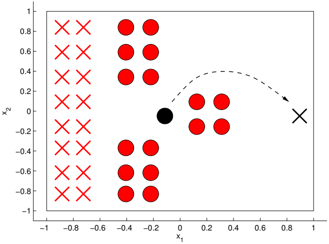

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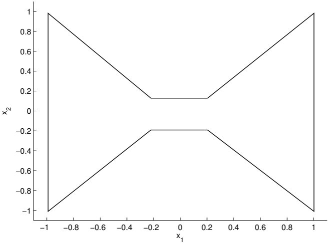

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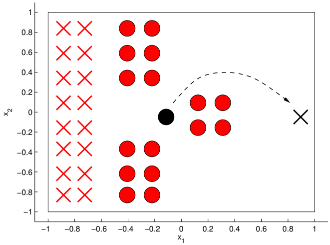

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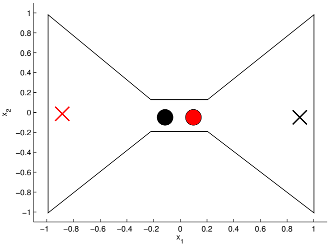

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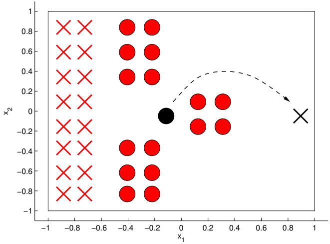

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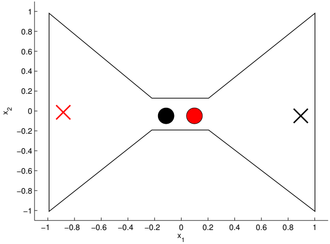

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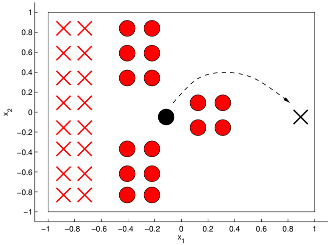

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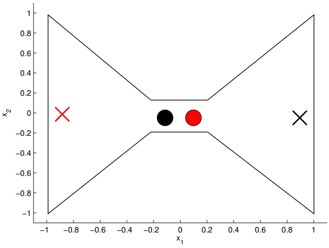

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- Even for large m, controllability may still be satisfied once (most of) the other subsystems are close to their equilibria $\rightsquigarrow$ application of small gain type arguments possible?
- In the bottleneck case, cooperation instead of simply avoiding each other will be needed. Can we tell self-resolvable from unresolvable deadlocks?

Discussion: hierarchical scheme

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- Is there a chance to obtain a truly parallel optimization with provable stability and feasibility???