Model Predictive Control: Robustness and Time Delays

Frank Allgöwer

Institute for Systems Theory and Automatic Control
University of Stuttgart

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Outline

1. Robust NMPC with Disturbance Invariant Sets
   - Problem Setup
   - Preliminaries
   - Robust NMPC Controller
   - Summary
2. MPC with Guaranteed Stability for Nonlinear Time-Delay Systems
   - Motivation
   - System Setup
   - NMPC for Time-Delay Systems
   - Three Schemes for Calculating Stabilizing Design Parameters
   - Summary

Problem Setup

Control problem

$\dot{x}(t) = f\big(x(t), u(t), w(t)\big),$ **nonlinear dynamics**

subject to

$x(t) \in \mathbb{X}, \quad u(t) \in \mathbb{U} \; \forall t \geq 0,$ **constraints**

$w(t) \in \mathbb{W} = \{w \in \mathbb{R}^{n_w} | \parallel w \parallel_\infty \leq w_{max}\},$ **bounded disturbances**

Goal: Robust MPC of nonlinear system with bounded disturbances

Existing schemes

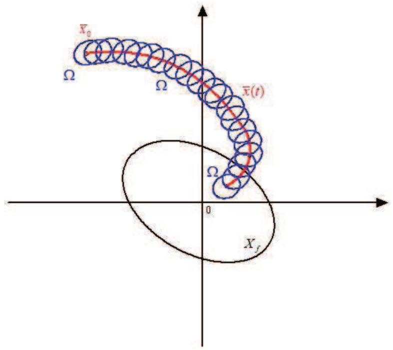
- Min-Max MPC $\rightarrow$ Computationally intractable
- MPC with contractive sets $\rightarrow$ Conservative solution

Suggested approach:
Nominal trajectory prediction + Robust invariant sets

Basic Idea

Error system

$$\dot{\bar{x}}(t) = f\big(\bar{x}(t), \bar{u}(t), 0\big),$$
$$v = x - \bar{x},$$
$$\dot{v} = f(x, u, w) - f(\bar{x}, \bar{u}, 0),$$



Controller structure

$$u = \bar{u} + \kappa(x, \bar{x}).$$

$\bar{u}$: **nominal input**,
$\kappa(x, \bar{x})$: **feedback law** renders the set Ω robustly invariant

- Nominal cost function
- Nominal prediction trajectory
- Actual traj. lie in Ω centered on nominal trajectory.

Problem to solve

- Calculation of Ω, $\kappa(x, \bar{x})$ based on **error system**
- Solution based on ISS

Input-to-state stability

Input-to-state stability (ISS)

The system is ISS, if there exists a function $E(v)$ such that

$$\alpha_1(\|v\|) \leq E(v) \leq \alpha_2(\|v\|), \quad (1a)$$

$$\frac{dE(v)}{dt} \leq -W(v), \|v\| \geq \rho(w_{max}), \quad (1b)$$

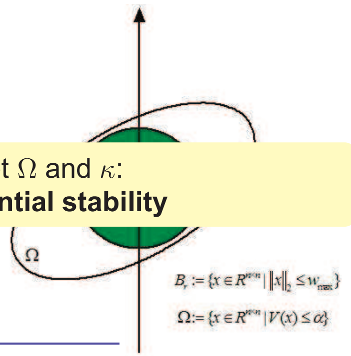
where $\alpha_i(\cdot) \in K_\infty$ and $\rho(\cdot) \in K$ and W positive definite

Let $E(x - \bar{x}) = E(v)$ with

Proposed concept to get Ω and κ:
Combine ISS and exponential stability

$\Downarrow$

Ω is robustly invariant



Exponential stability

Disturbance invariant sets using exponential stability

Let the function $E(v)$ with $\alpha_1(\|v\|) \leq E(v) \leq \alpha_2(\|v\|)$ and the scalars $\lambda > 0$ and $\mu > 0$ be such that

$$\frac{d}{dt}E(v(t)) + \lambda E(v(t)) - \mu w^T(t) w(t) \leq 0,$$

$$\Downarrow$$

$$\Omega := \{v \in R^{n_x} | E(v) \leq \frac{\mu w_{max}^2}{\lambda}\}$$

is a **disturbance invariant** set for the **error system**, i.e. $v(t) \in \Omega \ \forall t \geq t_0, \ w(t) \in \mathbb{W}$, if $v(t_0) \in \Omega$.

Remaining problem

Find κ and E

Suitably calculate

$$u = \kappa(x, \bar{x}) \qquad \text{and} \qquad E(v) = E(x - \bar{x})$$

such that the error system

$$\dot{v} = f(x, u, 0) - f(\bar{x}, u, w) \ [\neq f(\boldsymbol{v}, u, w) \text{ in general}]$$

is exponentially stable

Possible solution

- Backstepping
- Passivity-based control
- Linear differential inclusion [Yu et al., ACC,2010]
- ......
- Any suitable nonlinear controller design

Proposed robust NMPC controller

- Nominal prediction $\bar{u}$ such that nominal cost function is minimized
- Plus: auxiliary feedback law $\kappa(x, \bar{x})$
- Applied input: $u = \bar{u} + \kappa(x, \bar{x})$

Properties of the robust NMPC approach

Suppose that the NMPC optimization problem is feasible at time t_0.

- The MPC optimization problem is **feasible** at any time instant
- The closed-loop system is **robustly asymptotically ultimately bounded**
- The closed-loop system is **ISS** (w.r.t. $w(t)$)

Result is based on ISS

The **error** system controlled by $\kappa(x, \bar{x})$ is ISS

$\Downarrow$

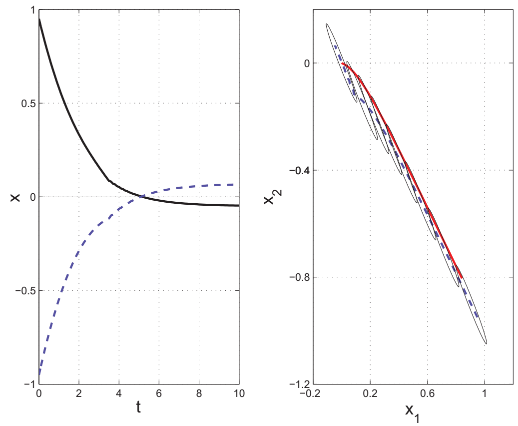
NMPC controlled **actual** system is ISS in its **whole feasible region**

$\Rightarrow$ Extension of the **local** ISS property to the **whole feasible region**!!!

Simulation example

Numerical example

$$\dot{x}_1 = 0.5x_1 + 0.15x_1^2 + x_2 + 0.6u$$
$$\dot{x}_2 = x_1 - 0.2x_2^2 + 0.6u + w$$



- Input constraints

$$-2 \leq u \leq 2,$$

- Disturbance bound

$$\|w\|_\infty \leq w_{max} = 0.1$$

- Weighting matrices

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$R = 1.$$

Summary

✓ Robust NMPC for systems with **bounded disturbances**

- Minimization of **nominal cost function**
- **Nominal input** plus **auxiliary** control law
- Auxiliary control law designed for **error system**

✗ Auxiliary controller design for error system in general hard task

✓ Actual system trajectories remain in **disturbance invariant set**

✓ Closed-loop system is **ultimately bounded** and **ISS**

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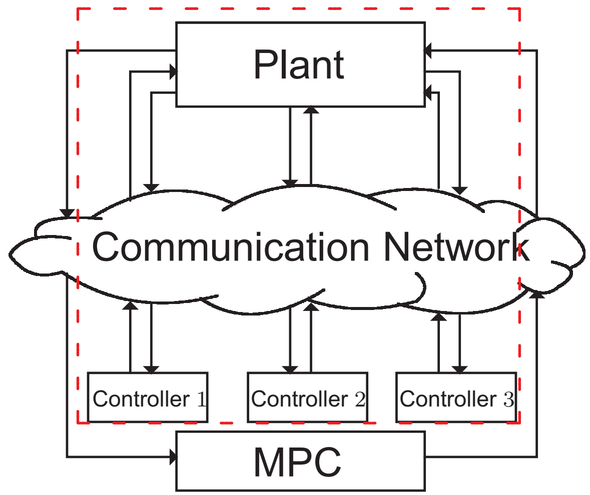
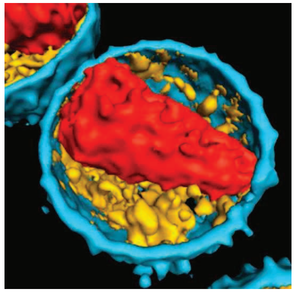
Motivation

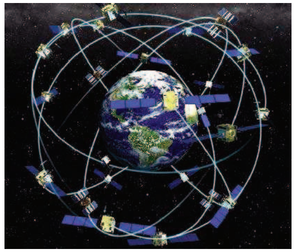


Other examples of time-delay systems

Chemical reactor model



HIV infection model



System with communication delays

System setup

Nonlinear time-delay system

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-\tau), u(t)) \\ x(\theta) &= \varphi(\theta), \qquad \forall \theta \in [-\tau, 0], \end{aligned}$$

- state $x_t \in \mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$
  defined by $x_t(s) = x(t+s), s \in [-\tau, 0]$
  $\rightarrow$ **infinite-dimensional** system
- input constraints $u(t) \in \mathcal{U} \subset \mathbb{R}^m$
- $f(0,0,0) = 0 \Rightarrow$ steady state at origin

Goal

- stabilize the origin
- achieve good performance

NMPC setup for time-delay systems

At each sampling instant t_i solve

$$\min_{u(\cdot)} J(x_{t_i}, u(\cdot)) = \int_{t_i}^{t_i+T} F(x(t'), u(t'))\, dt' + V(x_{t_i+T})$$

subject to

$$\begin{aligned} \dot{x}(t') &= f(x(t'), x(t'-\tau), u(t')) \\ u(t') &\in \mathcal{U} \\ x_{t_i+T} &\in \Omega_\tau \subseteq \mathcal{C}_\tau . \end{aligned}$$

Optimal solution $J^*(x_t)$ for $u^*(\cdot; x_t)$.

Control input according to the receding horizon strategy

$$u(t) = u^*(t; x_{t_i}), \quad t_i \le t \le t_i + \Delta .$$

Conditions for asymptotic stability

Theorem

Assume the following conditions are satisfied.

(a) The open loop finite horizon problem admits a **feasible solution** at initial time $t = 0$.

(b) For the nonlinear time-delay system $\dot{x}(t) = f(x(t), x(t-\tau), u(t))$, there exists a **locally asymptotically stabilizing** controller $u(t) = k(x_t)$ such that

  (i) $\forall x_t \in \Omega_\tau : u(t) = k(x_t) \in \mathcal{U}$
  (ii) the terminal region Ω_τ is positively invariant and
  (iii) $\forall x_t \in \Omega_\tau : \dot{V}(x_t) \le -F(x(t), k(x_t))$.

Then, the closed-loop system using MPC is asymptotically stable.

NMPC for time-delay systems

First result:

Setup and stability conditions are similar to the delay-free case.

Question:

How to get stabilizing design parameters?

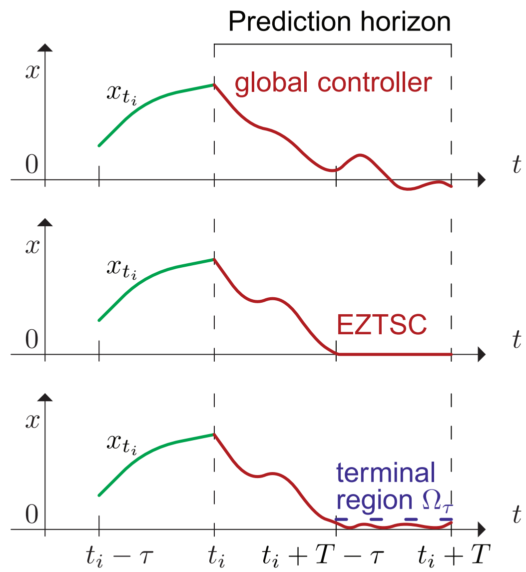
- How to define an appropriate terminal region $\Omega_\tau \subseteq \mathcal{C}_\tau$?
- How to obtain a local controller $k(\cdot)$?
- How to calculate the terminal cost function(al) V?

Overview of different schemes

Kwon, Lee and Han, 2001, 2002

Raff, Angrick, Findeisen, Kim and Allgöwer, 2007

“New”: Quasi-infinite horizon scheme



Basic idea

Quasi-infinite horizon

- Consider **Jacobi linearization**

$$\begin{aligned}\bar{\Sigma} : \dot{\bar{x}}(t) &= A\bar{x}(t) + A_\tau \bar{x}(t-\tau) + Bu(t)\\ \Sigma : \dot{x}(t) &= Ax(t) + A_\tau x(t-\tau) + Bu(t) + \underbrace{\Phi(x(t), x(t-\tau), u(t))}_{\text{higher order terms}}\end{aligned}$$

- Choose quadratic stage cost

$$F(x(t), u(t)) = x(t)^T Q x(t) + u(t)^T R u(t)$$

- Calculate linear controller $u(t) = k(x_t)$ for linearized system $\bar{\Sigma}$

- Determine a region Ω_τ such that for nonlinear system Σ
  - (i) Ω_τ is positively invariant
  - (ii) $\forall x_t \in \Omega_\tau : \dot{V} \le -F(x_t, k(x_t))$
  - (iii) $\forall x_t \in \Omega_\tau : |k(x_t)| \in \mathcal{U}$

Quasi-infinite horizon for delay-free systems

Delay-free systems

- Lyapunov function $V(x) = x^T P x$
- Local controller $u(t) = Kx(t)$
- Define terminal region using level set

$$\Omega_\tau = \{x \in \mathbb{R}^n : V(x) = x^T P x \le \alpha\}$$

- By choosing $\alpha > 0$ small enough, it is possible to guarantee
  - (iii) $\forall x \in \Omega_\tau : |Kx| \in \mathcal{U}$
  - (ii) $\forall x \in \Omega_\tau : \dot{V} \le -F(x, Kx)$
    - possible because Φ consists of only higher order terms
  - (i) Ω_τ is positively invariant due to (ii)

Definition of terminal region using level sets is not possible in infinite-dimensional case!

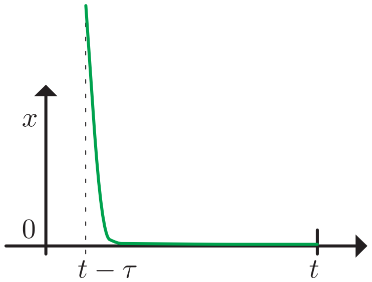
Why is such a definition not useful?

Terminal region

$$\Omega_\tau = \left\{ x_t \in \mathcal{C}_\tau \,:\, V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Sx(s)\,ds \leq \alpha \right\}$$

For arbitrarily small $\alpha > 0$,

- $x(t-\tau)$ could be arbitrarily large
- condition (ii) $\forall x_t \in \Omega_\tau \,:\, \dot{V} \leq -F(x, Kx)$ cannot be guaranteed!
- set invariance condition (i) cannot be guaranteed!



Quasi-infinite horizon for time-delay systems

Possible schemes

- Combination of Lyapunov-Krasovskii and Lyapunov-Razumikhin [CDC 2009]
- Lyapunov-Krasovskii arguments and norm bounds [accepted at IFAC TDS 2010]
- Lyapunov-Razumikhin [submitted to CDC 2010]

Brief overview on results

Krasovskii condition plus norm bound

- only **Krasovskii** condition on local control law
- more complicated terminal region $V(x_t) \leq \frac{\beta\,\alpha^2}{4}, \quad \|x_t\|_\tau \leq \frac{\alpha}{2}$
- **Krasovskii** functional V

Combination of Krasovskii and Razumikhin

- **Razumikhin** condition on local control law
- **simple** terminal region $\max_{\theta\in[-\tau,0]} x(t+\theta)^T Px(t+\theta) \leq \alpha$
- **Krasovskii** functional V

Razumikhin condition

- **Razumikhin** condition on local control law
- **simple** terminal region $\max_{\theta\in[-\tau,0]} x(t+\theta)^T Px(t+\theta) \leq \alpha$
- $V = \max_{\theta\in[-\tau,0]} x(t+\theta)^T Px(t+\theta)$

Summary

- ✔ Derivation of a 'finite' terminal region for MPC of nonlinear time-delay systems
- ✔ Three schemes based on Jacobi-linearization
- ✔ Each scheme contains delay-free case as special case
- ✘ Additional arguments necessary compared to delay-free case
- ✔ Shorter prediction horizon than for EZTSC
- ✔ Only locally stabilizing control law necessary

Conclusions

Summary

1. Robust NMPC for systems with bounded disturbances
   - prediction of nominal trajectories
   - disturbance invariant sets
   - auxiliary control law
   - ISS and exponential stability of error system
2. NMPC for time-delay systems
   - terminal region in infinite-dimensional space
   - calculation using Jacobi-linearization
   - different possible extensions of delay-free results

Future work

How can the presented results be applied to distributed NMPC?

- uncertain neighbour information → ISS
- communication delays

Acknowledgements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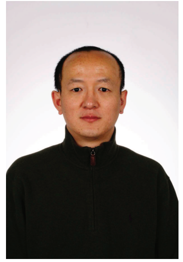
Shuyou Yu


Institute for Systems Theory and Automatic Control