Logistics, queueing networks and model reduction

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LCCC 2010
Lund, May 19–21, 2010.

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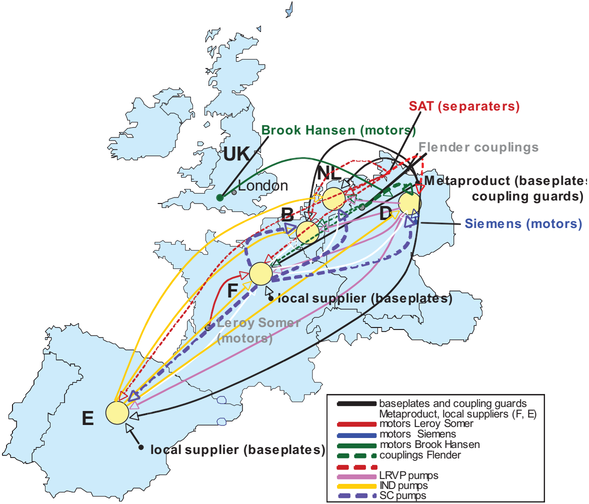
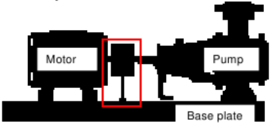
Motivation

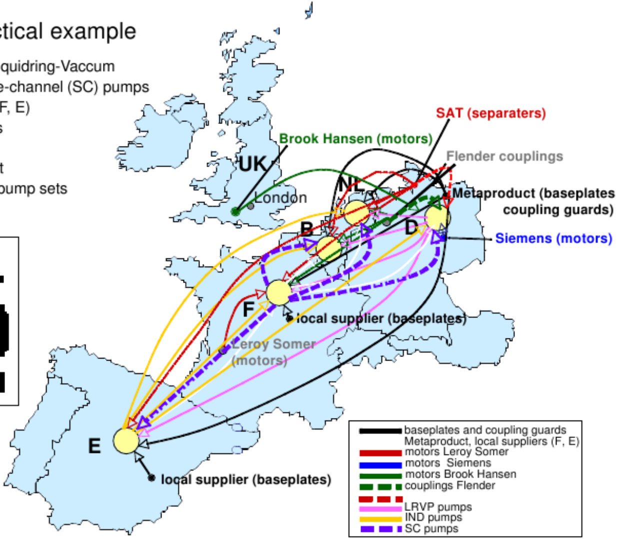

SAT (separaters)
Brook Hansen (motors)
Flender couplings
UK
London
NL
Metaproduct (baseplates
coupling guards)
B
D
Siemens (motors)
F
local supplier (baseplates)
Leroy Somer
(motors)
E
local supplier (baseplates)
baseplates and coupling guards
Metaproduct, local suppliers (F, E)
motors Leroy Somer
motors Siemens
motors Brook Hansen
couplings Flender
LRVP pumps
IND pumps
SC pumps

Characteristics of the practical example

- 3 production sites (D, F, E) for Liquidring-Vaccum (LRVP), industrial (IND) and side-channel (SC) pumps
- 5 distribution centres (D, NL, B, F, E)
- 33 first and second-tier suppliers for the production of pumps
- 90 suppliers for components that are needed for the assembly of pump sets
- More than 1000 customers

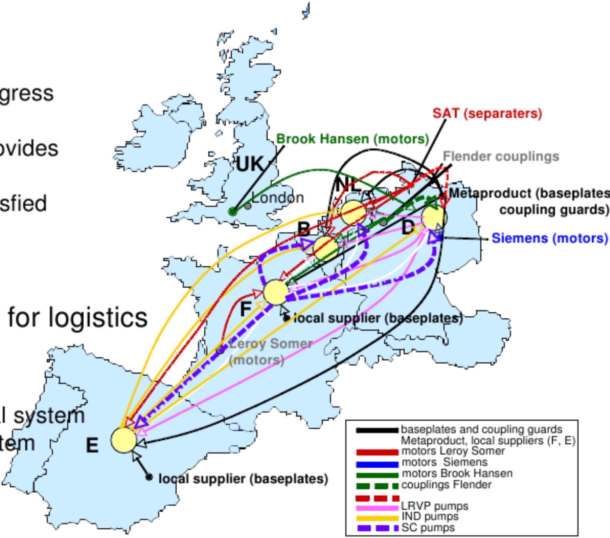
Pump Set





Instability leads to

- Unbounded work in progress
- High inventory costs
- Standard simulation provides little information
- Large number of unsatisfied orders
- Loss of customers



Mathematical models for logistics networks

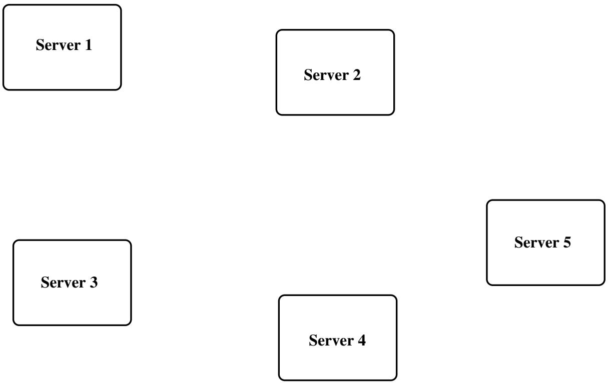
- Continuous dynamical system
- Hybrid dynamical system
- Queueing networks
- Fluid networks

Queueing Systems

Queueing systems provide a stochastic framework for the modelling of logistic systems

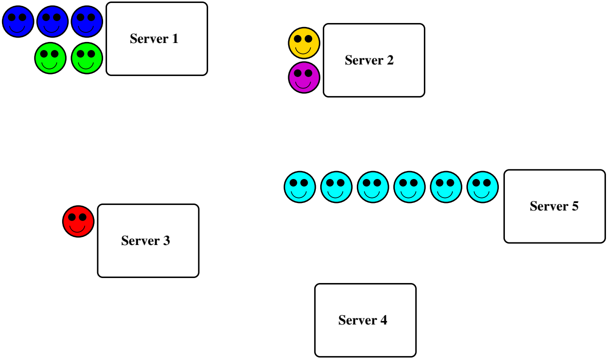
Queueing Systems

Consider a set of servers which are able to treat different classes of jobs.



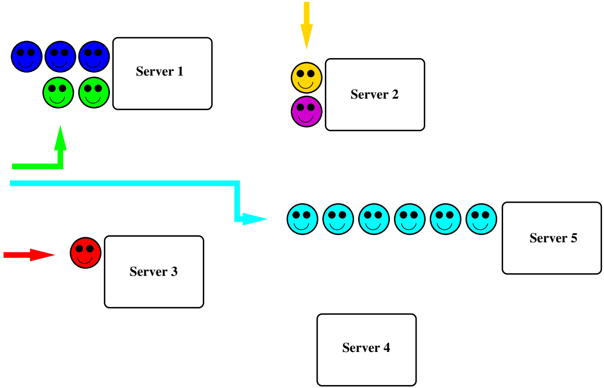
Queueing Systems

Without loss of generality each class only receives service at one given server. Unserved jobs wait in a queue.



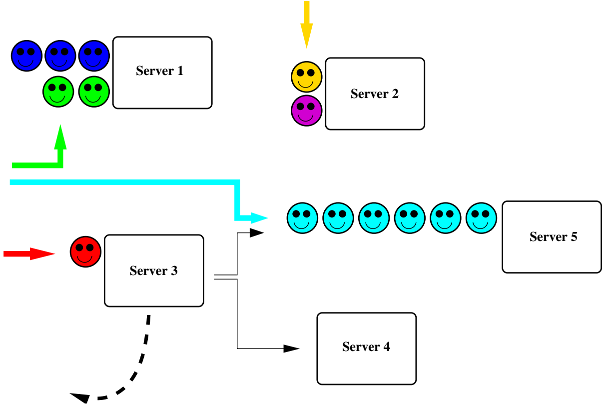
Queueing Systems

In open systems jobs arrive from the outside according to some stochastic process.



Queueing Systems

After service jobs leave the network or with a certain probability they go to another station to receive service there.



Queueing Systems - The Maths

Classes $k = 1, \dots, K$.

Interarrival times: $\xi_k(n), n = 0, 1, 2, \dots$ i.i.d. $\mathbb{E}(\xi_k(0)) < \infty$

Service times: $\eta_k(n), n = 0, 1, 2, \dots$ i.i.d. $\mathbb{E}(\mu_k(0)) < \infty$

routing matrix $P = (p_{ij})$,
p_{ij} - probability that job of class i becomes a job of class j
Assumption: $r(P) < 1$.

Queueing Systems - Balance Equations

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Assumption: $r(P) < 1$.

$$Q(t) = Q(0) + A(t) + P^T S(t) - S(t)$$

with

$$A_j(t) = \max \left\{ n \mid \sum_{m=0}^{n} \xi_k(m) \leq t \right\}$$

$S(t)$ is the service process - depends on the service discipline.

The state space $\mathbb{X}$ is very often countable, but also depends on the service discipline.

Stability of Queueing Systems

Definition A queueing system is called stable if it is Harris recurrent.

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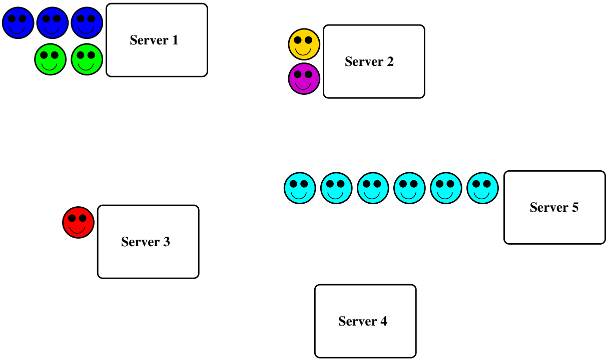
Technicalities aside, Harris recurrence means that there is an attractive invariant measure π for the Markov process.
Here invariant means that for all $t > 0$ and all measurable sets A

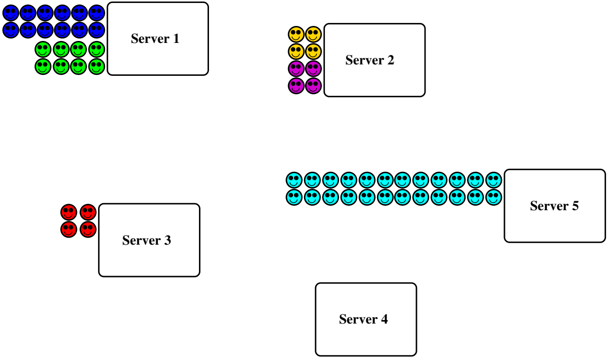
$$\pi(A) = \int_{\mathbb{X}} P_t(x, A)\pi(dx)\,,$$

where $P_t(x, B)$ is the probability to go from x to the set B in time t.
In the long run, the probability of being in a set X is $\pi(X)$.

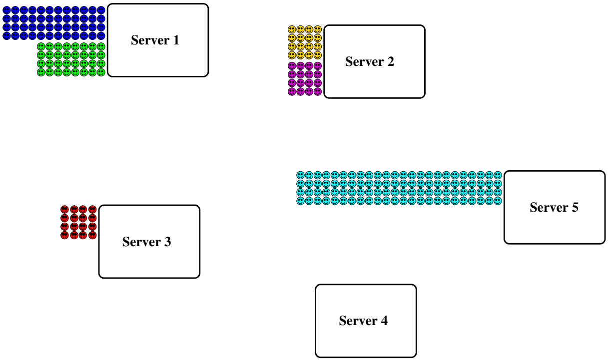
Queueing Systems - Fluid Limits



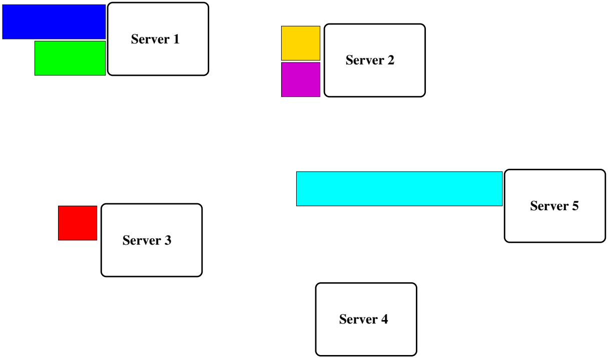
Queueing Systems - Fluid Limits



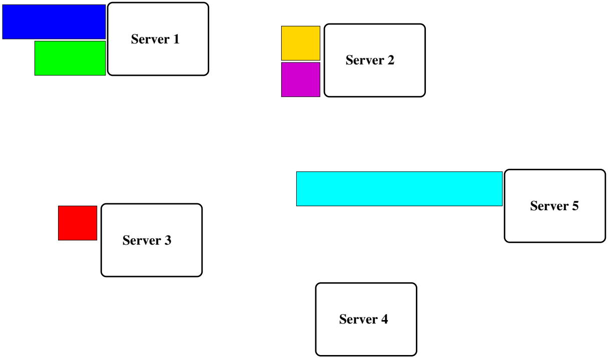
Queueing Systems - Fluid Limits



Queueing Systems - Fluid Limits



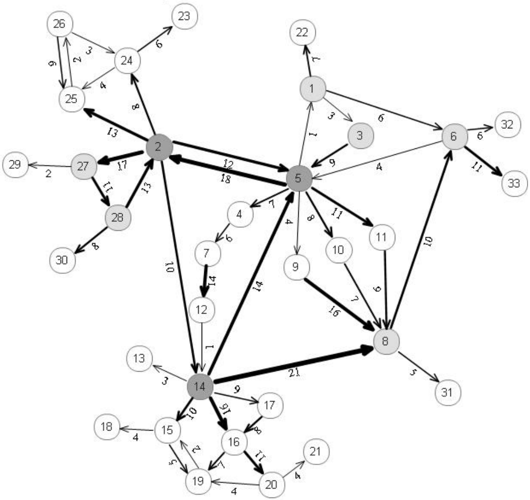
Queueing Systems - Fluid Limits



Metatheorem (Rybko/Stolyar 1992, Dai 1995) If the fluid limit model is stable at 0, then the corresponding queueing system is Harris recurrent.

Ranking in Graphs

Ranking schemes try to extract information about the importance/relevance of a vertex from graph properties.



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- Colums are rescaled to have column sum 1 (where possible).
- A is made colum stochastic by adding artificial entries in zero columns.
- A is made irreducible e.g. by considering for some $\alpha \in (0,1)$

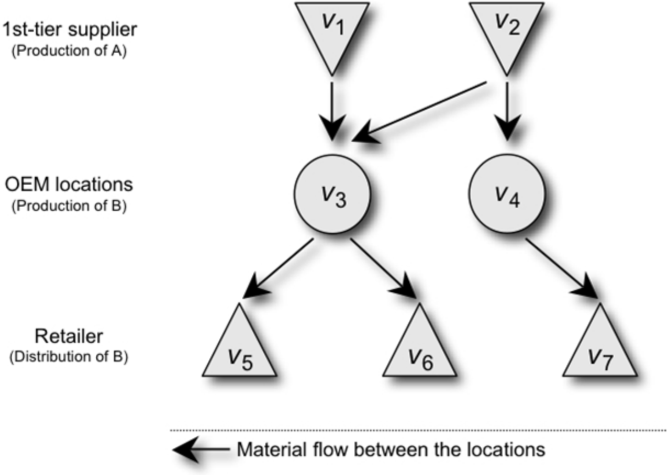
$$\tilde{A} := \alpha A + (1-\alpha)\mathbf{e}\mathbf{e}^T$$

Then Perron-Frobenius theory says that there is an eigenvector $r > 0$ such that

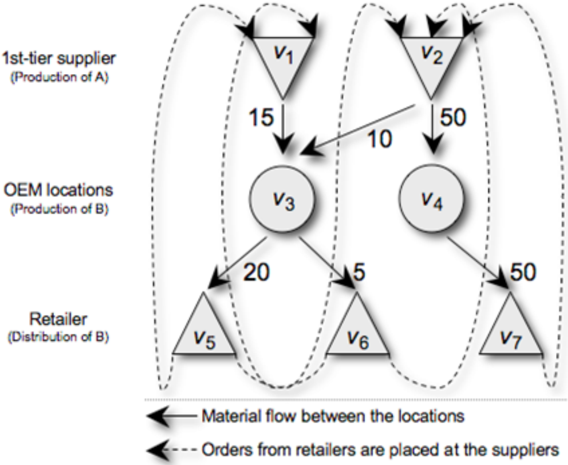
$$Ar = r$$

The entries of r quantify the importance of the nodes.

Ranking in Graphs: Eliminating zero columns



Ranking in Graphs: Eliminating zero columns



Ranking in Graphs - Ensuring irreducibility

Given the weighted adjecency matrix $A \in \mathbb{R}^{n \times n}$ consider the enlarged matrix

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

Ranking in Graphs

Consider the graph described by A as weakly coupled with a larger network. Coupling described by vectors v and w.

$$B = \begin{bmatrix} \alpha A + (1-\alpha) v_n \mathbf{e}_n^T & w_n \mathbf{e}_m^T \\ (1-\alpha) v_m \mathbf{e}_n^T & w_m \mathbf{e}_m^T \end{bmatrix}$$

Notation: $\mathbf{e} := \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$

Ranking in Graphs

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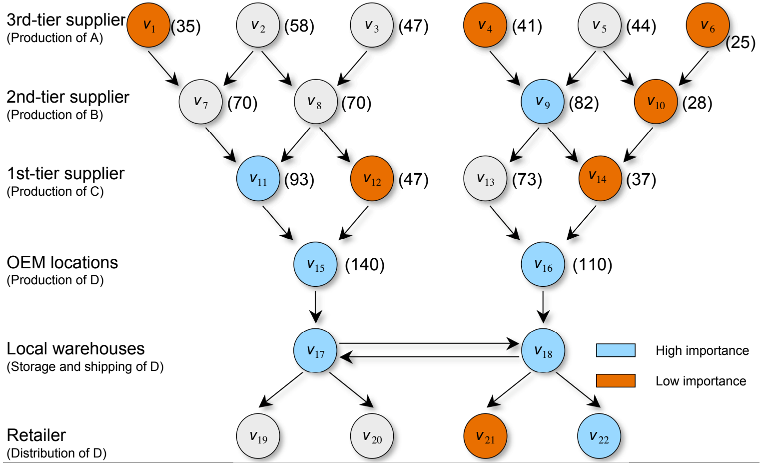
Proposition Assume that B is irreducible, then if $x = \begin{bmatrix} x_n^T & x_m^T \end{bmatrix}^T$ is an eigenvector corresponding to the eigenvalue 1 of B then x_n is an eigenvector corresponding to the eigenvalue 1 of the matrix

$$A_\alpha(v, w) := \alpha A + (1-\alpha) \left(v_n + \frac{\mathbf{e}_m^T v_m}{1 - \mathbf{e}_m^T w_m} w_n \right) \mathbf{e}_n^T .$$

Furthermore, $A_\alpha(v, w)$ is irreducible.

A simple example



Ranking in Queueing Networks

How can we evaluate if a reduced order model is sufficiently close to the original one ?

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The invariant probability distributions should be close to the original one.

Particular case: Jackson Networks

In Jackson networks each server serves exactly one class of jobs. The arrival process is Poisson and the service times are exponentially distributed.

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In Jackson networks each server serves exactly one class of jobs.
The arrival process is Poisson and the service times are exponentially distributed.
Vector of external arrivals: α
Traffic equation for effective load of a server:

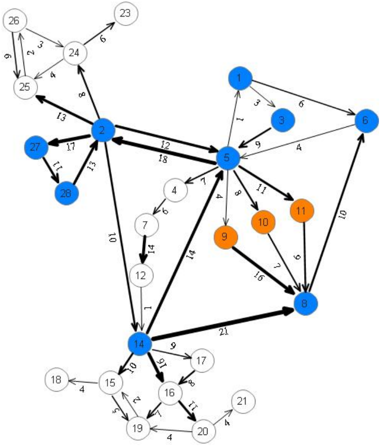
$$\lambda = P^T \lambda + \alpha \, .$$

Without the embedding in a larger network λ is the ranking vector.
λ determines the stationary probability distribution. So in this case there is no problem.

Heuristics

Theorem The following procedures for reduction do not change the rank of unaffected nodes.

Heuristics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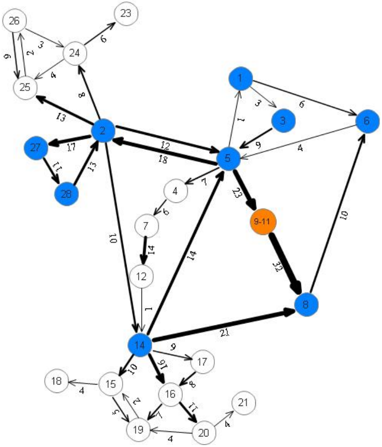
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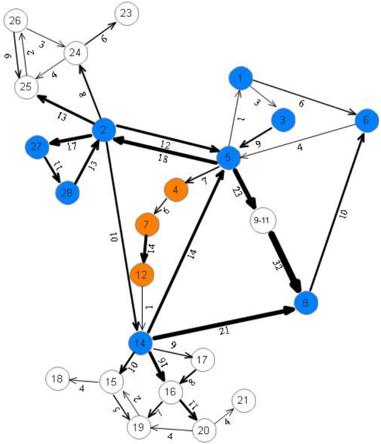
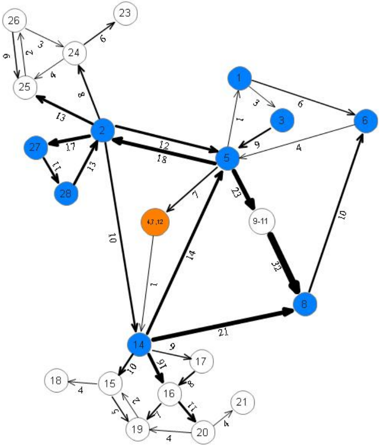
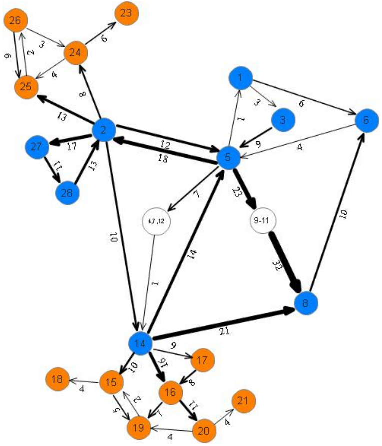
Heuristics





Heuristics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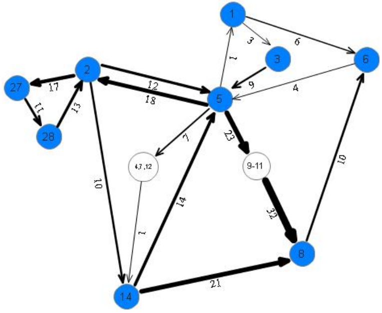
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Thank you