# Sets in Excess Demand in Ascending Auctions with Unit-Demand Bidders

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# Outline of the Talk

- The Vickrey second-price auction
- Generalizations
- The DGS auction mechanism
- Problems with DGS
- Sets in excess demand
- A new auction mechanism
- Theoretical results and some (preliminary) simulation results
- Conclusion

# Vickrey auctions (1/2)

- A Vickrey auction is a sealed-bid auction where bidders submit written bids without knowing the bid of the other bidders
- The highest bidder wins, but the price paid is the second-highest bid

**Example**

Two bidders (1 and 2) and a single item (the sellers reservation price is set to 0 for simplicity). The valuation of the item for bidder $i = 1, 2$ is given by $v_i$ where $v_1 = \$10$ and $v_2 = \$2$. If both bidders bid truthfully, bidder 1 wins the auction and pay \$2 for the item.

- Note that the set of equilibrium prices in the above example is given by the interval $[v_2, v_1] = [2, 10]$ and that the "minimum" equilibrium price is selected by the Vickrey mechanism.

# Vickrey auctions (1/2)

- The Vickrey auction mechanism has a number of striking properties:
  - Is ex post efficient, i.e., the winner is the bidder with the highest valuation of the item
  - It is a (weakly) dominant strategy for the bidders to bid their true valuation, i.e., the mechanism is strategy-proof
  - It is individually rational
- These properties hold in a generalized Vickrey auction with unit-demand bidders with $m$ homogeneous items where the $m$ bidders with highest valuations of the goods receive one item each and pay the $(m+1)$-th valuation
- (Note: the result does not hold for multi-demand bidders)

# Generalizations (1/3)

- What if the goods are heterogeneous (but bidders still wish to acquire at most one item)?
- In a market with $n$ unit-demand buyers and $m$ objects, Demange and Gale (1985, Econometrica) demonstrated that the set of equilibrium prices forms a complete lattice
- Thus, the set of equilibrium prices contains a "maximum" and a "minimum" equilibrium price vector.
- If an auction mechanism selects the "minimum equilibrium prices" no buyer or no group of buyers can gain by strategic misrepresentation (strategy-proofness) and the outcome is efficient and individual rational
- Such an auction mechanism is a generalization of the Vickrey auction

# Generalizations (2/3)

## Example

Suppose that $B = \{1, 2\}$, $I = \{1, 2\}$ and that the sellers reservation price is $r_i = 0$ for all $i \in I$. The values of the items to the bidders are given by the matrix:

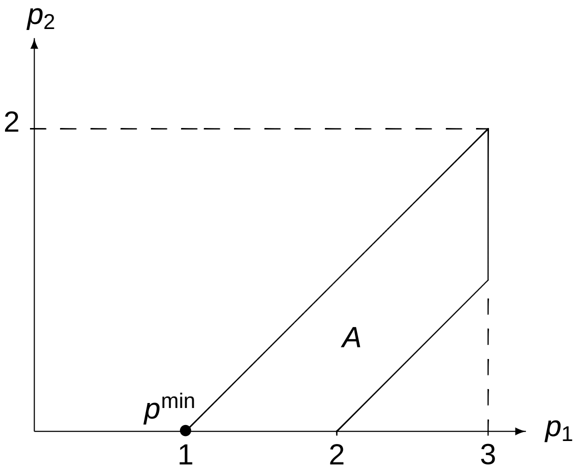
$$V = \{v_{bi}\} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

Clearly, the equilibrium prices must satisfy:

$$\begin{aligned} 0 \leq p_1 &\leq 3 \\ 0 \leq p_2 &\leq 3 \\ 1 \leq p_1 - p_2 &\leq 2 \end{aligned}$$

# Generalizations (3/3)

The set of equilibrium prices is given by the area $A$



# The DGS algorithm (1/4)

- The auction mechanism from Demange and Gale assumes that all bidders report their valuation for each of the $m$ objects
- However, bidders often fear complete revelation of information, as emphasized in recent years by e.g. Ausubel (2004, Am. Ec. Rev.) and Perry and Reny (2005, Rev. Ec. Stud.)
- Thus, a partially privacy preserving mechanism (in the sense that bidders need not report their valuations of all items) is preferable

# The DGS algorithm (2/4)

- Such a mechanism was proposed by Demange et al. (1986, J. Pol. Ec.)
- To give a formal description of the DGS auction algorithm, two basic definitions are necessary

**Definition**

A set of items $S$ is said to be overdemanded at prices $p$ if the number of bidders demanding only items in $S$ is greater than the number of items in the set.

**Definition**

A minimal overdemanded set of items is an overdemanded set of items where no proper subset is overdemanded.

# The DGS algorithm (3/4)

## Example

Suppose that $B = \{1, 2, 3, 4, 5\}$ and $I = \{1, 2, 3\}$. The values of the items to the bidders are given by the matrix:

$$V = \{v_{bi}\} = \begin{bmatrix} 0 & 52 & 12 \\ 39 & 39 & 0 \\ 10 & 0 & 89 \\ 98 & 20 & 99 \\ 67 & 57 & 38 \end{bmatrix}$$

At the reservation prices $r = 0$ the bidders' initial demand sets become $D_1(r) = \{2\}$, $D_2(r) = \{1, 2\}$, $D_3(r) = \{3\}$, $D_4(r) = \{3\}$, and $D_5(r) = \{1\}$. Thus:

$$OD(r) = \{\{3\}, \{1, 2\}, \{1, 2, 3\}\},$$
$$MOD(r) = \{\{3\}, \{1, 2\}\}.$$

# The DGS algorithm (4/4)

## DGS Algorithm

Introduce an iteration counter $t$ and let $p^t$ denote the price vector in iteration $t$. Set $t := 0$ and initialize the price vector to the reservation prices, $p^0 := 0$. For each iteration $t = 0, 1, 2, \ldots$:

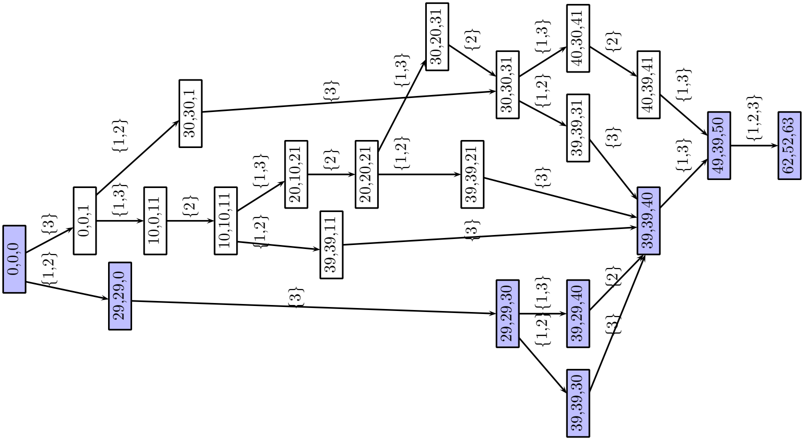
1. Collect the demand sets $D_b(p^t)$ of every bidder $b \in B$ at the current prices $p^t$.
2. If there is no overdemanded set of items for the given $D_b(p^t)$ the algorithm is terminated. Otherwise:
3. Choose a minimal overdemanded set of items $S^t$.
4. Compute the updated price vector $p^{t+1}$ whose elements are given by

$$p_i^{t+1} = \begin{cases} p_i^t + k & \text{if } i \in S^t, \\ p_i^t & \text{otherwise.} \end{cases}$$

# Problems with DGS (1/5)

- There are typically more than one minimal overdemanded set. Thus, the auctioneer is confronted with a choice of which prices to update for the next iteration.
- DGS typically requires a large number of iterations (i.e. price updates) before it converges. Thus, from a usability perspective, DGS in not very practical since it may be both costly and time consuming for the bidders to participate in it.
- If both sets $S$ and $T$ are minimal overdemanded, it is obvious that the prices for all items in the set $S \cup T$ must be raised in order to reach an equilibrium. However, DGS does not allow for simultaneous price increases in $S$ and $T$.

# Problems with DGS (2/5)

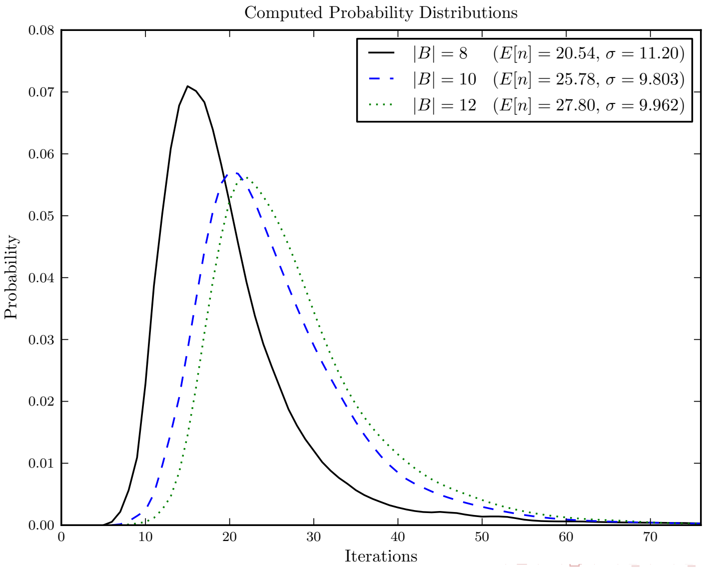


# Problems with DGS (3/5)

Table: Summary of graph orders and sizes for the three cases $(|B|, |I|) \in (\{8, 10, 12\}, \{5\})$ computed from 500 sample graphs per case.

| Vertices | | | | Arcs | | | |
|---|---|---|---|---|---|---|---|
| $\|B\|$ | $\|V\|_{min}$ | $\|V\|_{max}$ | $\|V\|_{mean}$ | $\|B\|$ | $\|A\|_{min}$ | $\|A\|_{max}$ | $\|A\|_{mean}$ |
| 8 | 27 | $2.372 \cdot 10^4$ | $2.265 \cdot 10^3$ | 8 | 38 | $6.704 \cdot 10^4$ | $5.181 \cdot 10^3$ |
| 10 | 148 | $8.426 \cdot 10^5$ | $9.037 \cdot 10^4$ | 10 | 286 | $3.023 \cdot 10^6$ | $2.156 \cdot 10^5$ |
| 12 | 433 | $2.401 \cdot 10^6$ | $3.115 \cdot 10^5$ | 12 | 944 | $9.052 \cdot 10^6$ | $8.093 \cdot 10^5$ |

# Problems with DGS (4/5)



# Problems with DGS (5/5)

Table: Minimum and maximum iterations needed for DGS to find equilibria for the three cases $(|B|, |I|) \in (\{8, 10, 12\}, \{5\})$ computed from 500 sample graphs per case. $n^-$ and $n^+$ denote the number of iterations needed for the best and worst path through a graph respectively.

| Minimum Iterations ($n^-$) | | | | | Maximum Iterations ($n^+$) | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\|B\|$ | $n^-_{min}$ | $n^-_{max}$ | $n^-_{mean}$ | $\sigma(n^-)$ | $\|B\|$ | $n^+_{min}$ | $n^+_{max}$ | $n^+_{mean}$ | $\sigma(n^+)$ |
| 8 | 6 | 82 | 12.62 | 5.164 | 8 | 11 | 178 | 58.28 | 36.71 |
| 10 | 7 | 29 | 14.09 | 3.217 | 10 | 17 | 320 | 90.09 | 55.93 |
| 12 | 8 | 27 | 14.75 | 2.775 | 12 | 22 | 328 | 84.81 | 49.91 |

# Sets in Excess Demand (1/3)

- To define a new algorithm (that hopefully will perform better than DGS) the following is useful:

**Definition**

All bidders that demand some item in the set $S$ at prices $p$ are collected in the set $U(S, p)$. A set of items $S$ is weakly underdemanded at prices $p$ if $S \subseteq I^{+}(p)$ and $\#U(S, p) \leq \#S$.

**Theorem (Mishra and Talman, Games Ec. Behav., 2009)**

The price vector $p$ equals $p^{\text{min}}$ if and only if there are no overdemanded sets of items and no weakly underdemanded sets of items at prices $p$.

# Sets in Excess Demand (2/3)

- We are now ready to provide a weakening of the definition of minimal overdemanded sets and introduce the notion of sets in excess demand

**Definition**

A set of items $S$ is in excess demand at prices $p$ if at prices $p$ the set $S$ is overdemanded and no subset $T$ of $S$ is weakly underdemanded by the bidders that only demand items in $S$.

**Theorem**

If $S \in MOD(p)$ then $S \in ED(p)$ and, as a consequence, $MOD(p) \subseteq ED(p)$.

# Sets in Excess Demand (3/3)

**Theorem**

If $S \in ED(p)$ and $T \in ED(p)$ then $S \cup T \in ED(p)$.

- We have the following corollary to the above theorem which states that there exists a unique "largest" set in excess demand at given prices

**Corollary**

If $ED(p) \neq \varnothing$ then there exists a unique set $S_* \in ED(p)$ where $\#S_* > \#T$ for each $T \in ED(p) - \{S_*\}$.

- Thus, by basing the new algorithm on the "largest" set, there is a unique price increment in each step of the algorithm

# The New Algorithm (1/6)

## The new algorithm

The new algorithm replaces Steps 3 and 4 of DGS with:

3. Identify the unique set in excess demand with maximal cardinality.
4. Compute the updated price vector $p^{t+1}$ whose elements are given by

$$p_i^{t+1} = \begin{cases} p_i^t + k_{ud} & \text{if } i \in S^t, \\ p_i^t & \text{otherwise.} \end{cases}$$

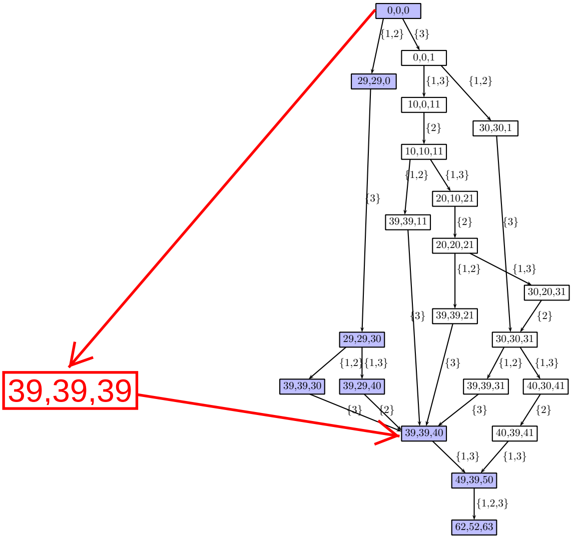
# The New Algorithm (2/6)

**Theorem**

The new algorithm converges at the minimum equilibrium prices $p^{\text{min}}$ in a finite number of iterations.

- We next provide some numerical results based on a (preliminary) simulation study
- But first, recall that in the previous numerical example DGS terminated in 7-10 iterations. The new algorithm requires 5 iterations.

# The New Algorithm (3/6)



# The New Algorithm (4/6)

- Another example (a little bit more complex graph but still very small)
- $B = \{1, 2, 3, 4, 5\}$ and $I = \{1, 2, 3\}$ (again reservation prices $r = 0$).
- The values of the items to the bidders are given by the matrix:

$$V = \{v_{bi}\} = \begin{bmatrix} 99 & 29 & 60 \\ 57 & 75 & 57 \\ 0 & 70 & 29 \\ 78 & 86 & 55 \\ 0 & 23 & 46 \end{bmatrix}$$

- DGS fastest path 11 iterations, longest path 14 iterations
- New algorithm 5 iterations

# The New Algorithm (5/6)

# The New Algorithm (6/6)

- We only have preliminary simulation results for small auctions (5 bidders and 3 objects):

Table: Minimum iterations needed for the algorithms to find equilibria for the case $(|B|, |I|) = (\{5\}, \{3\})$. $\bar{n}_{\text{mean}}$ denotes the mean number of iterations needed for the best path through a graph.

| Auction type | $\bar{n}_{\text{mean}}$ | Expected iterations |
|---|---|---|
| New | 4.54 | 4.54 |
| DGS | 9.95 | 16.71 |

- Note: in each simulation, the number of iterations required for the new algorithm to converge was strictly lower than the number of required iterations in DGS independently of how the MODs were chosen.

# Conclusions

- There are reasons to believe that DGS can be improved (and there is not reason not to improve it)
- The paper defined a new algorithm for auction markets with heterogeneous items and unit-demand bidders
- This algorithm converges in a finite number of iterations and preliminary simulation results demonstrate that this algorithm indeed is faster than DGS

THANKS!