Games and Price Mechanisms for Distributed Control

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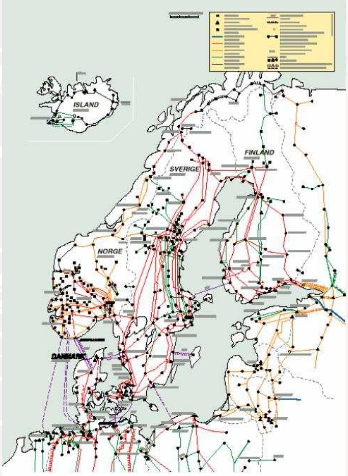
Needs for distributed control theory

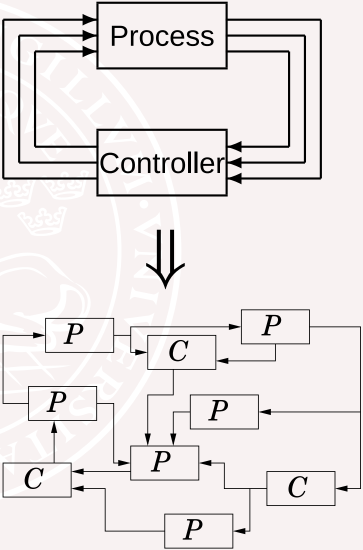
Three major challenges:

- Rapidly increasing complexity
- Dynamic interaction
- Information is decentralized



Building theoretical foundations for distributed control

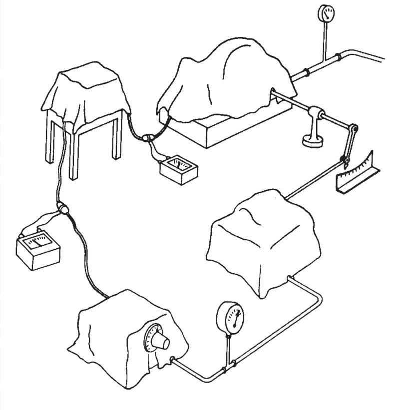
A centralized paradigm dominates theory and curriculum today



We need methodology for

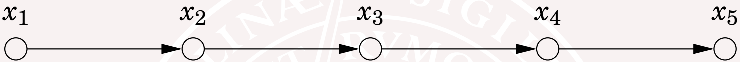
- Decentralized specifications
- Decentralized design
- Validation of global behavior

Can Systems be Certified Distributively?



Componentwise performance verification without global model?

Example 1: A vehicle formation

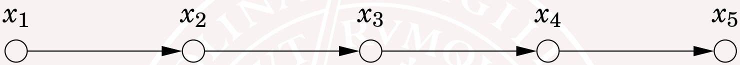


Each vehicle obeys the independent dynamics

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1u_1(t) + w_1(t) \\ B_2u_2(t) + w_2(t) \\ B_3u_3(t) + w_3(t) \\ B_4u_4(t) + w_4(t) \end{bmatrix}$$

The objective is to make $\mathbf{E}|Cx_{i+1} - Cx_i|^2$ small for $i = 1, \ldots, 4$.

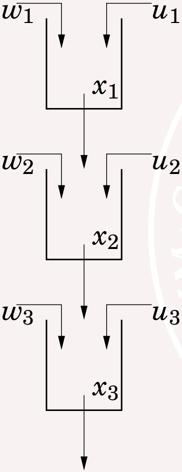
Example 2: A supply chain for fresh products



Fresh products degrade with time:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} -u_1(t) + w_1(t) \\ u_1(t) - u_2(t) \\ u_2(t) - u_3(t) \\ u_3(t) + w_4(t) \end{bmatrix}$$

Example 3: Water distribution systems

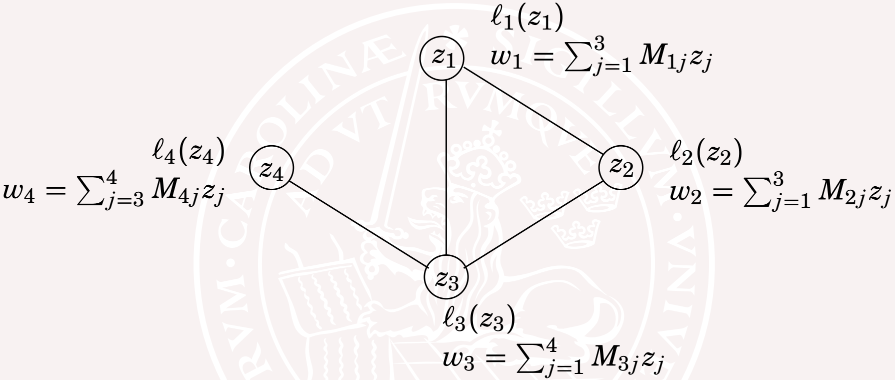


$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1 u_1 + w_1 \\ B_2 u_2 + w_2 \\ B_3 u_3 + w_3 \\ B_4 u_4 + w_4 \end{bmatrix}$$

Example 4: Wind farms

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1 u_1(t) + w_1(t) \\ B_2 u_2(t) + w_2(t) \\ B_3 u_3(t) + w_3(t) \\ B_4 u_4(t) + w_4(t) \end{bmatrix}$$

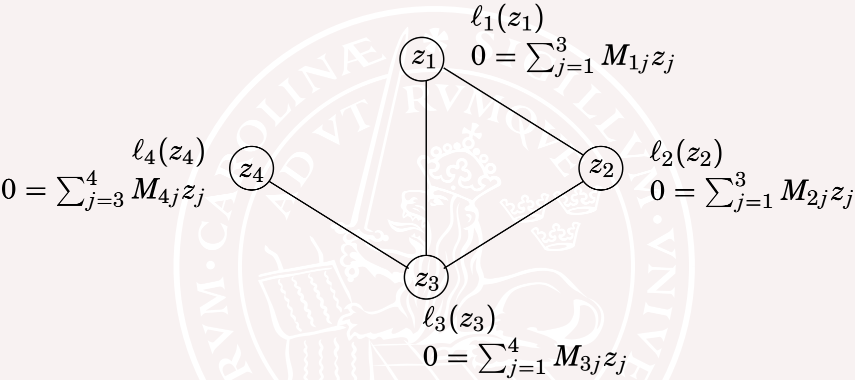
Disturbance Rejection with Graph Structure



Evaluate the cost subject to worst case disturbances w:

$$\max_w \sum_i \ell_i(z_i) \text{ where } \sum_j M_{ij} z_j = w_i$$

Optimization Without Disturbances



Minimize$_z$ $\sum_i \ell_i(z_i)$ subject to $\sum_j M_{ij} z_j = 0$

Assume ℓ_i convex. Game formulation by dual decomposition!

Outline

- Introduction
- **Games from Dual Decomposition**
- Game Formulation for Distributed Disturbance Rejection
- Dynamic Problems

50 year old idea: Dual decomposition

$$\min_{z_i}[V_1(z_1,z_2)+V_2(z_2)+V_3(z_3,z_2)]$$

$$=\max_{p_i}\min_{z_i,v_i}\Big[V_1(z_1,v_1)+V_2(z_2)+V_3(z_3,v_3)+p_1(z_2-v_1)+p_3(z_2-v_3)\Big]$$

The optimum is a Nash equilibrium of the following game:

The three computers try to minimize their respecive costs

Computer 1: $\min_{z_1,v_1}\Big[V_1(z_1,v_1)-p_1v_1\Big]$

Computer 2: $\min_{z_2}\Big[V_2(z_2)+(p_1+p_3)z_2\Big]$

Computer 3: $\min_{z_3,v_3}\Big[V_3(z_3,v_3)-p_3v_3\Big]$

while the ”market makers” try to maximize their payoffs

Between computer 1 and 2: $\max_{p_1}[p_1(z_2-v_1)]$

Between computer 2 and 3: $\max_{p_3}[p_3(z_2-v_3)]$

Decentralized Bounds on Suboptimality

Given any $p_1, p_3, \bar{z}_1, \bar{z}_2, \bar{z}_3$, the distributed test

$$V_1(\bar{z}_1, \bar{z}_2) - p_1\bar{z}_2 \leq \alpha \min_{z_1, v_1} [V_1(z_1, v_1) - p_1 v_1]$$

$$V_2(\bar{z}_2) + (p_1 + p_3)\bar{z}_2 \leq \alpha \min_{z_2} [V_2(z_2) + (p_1 + p_3)z_2]$$

$$V_3(\bar{z}_3, \bar{z}_2) - p_3\bar{z}_2 \leq \alpha \min_{z_3, v_3} [V_3(z_3, v_3) - p_3 v_3]$$

implies that the globally optimal cost J^* is bounded as

$$J^* \leq V_1(\bar{z}_1, \bar{z}_2) + V_2(\bar{z}_2) + V_3(\bar{z}_3, \bar{z}_2) \leq \alpha J^*$$

Proof: Add both sides up!

The saddle point algorithm

Update in gradient direction:

Computer 1:
$$\begin{cases} \dot{z}_1 = -\partial V_1/\partial z_1 \\ \dot{v}_1 = -\partial V_1/\partial z_2 + p_1 \end{cases}$$

Computer 1 and 2:
$$\dot{p}_1 = z_2 - v_1$$

Computer 2:
$$\dot{z}_2 = -\partial V_2/\partial z_2 - p_1 - p_3$$

Computer 2 and 3:
$$\dot{p}_3 = z_2 - v_3$$

Computer 3:
$$\begin{cases} \dot{z}_3 = -\partial V_3/\partial z_3 \\ \dot{v}_3 = -\partial V_3/\partial z_2 + p_3 \end{cases}$$

Globally convergent if V_i are convex!
[Arrow, Hurwicz, Usawa 1958]

Important Aspects of Dual Decomposition

- Very weak assumptions on graph
- No need for central coordination
- Decentralized bounds on suboptimality
- Unique Nash equilibrium corresponds to global optimum
- Natural learning procedure is globally convergent

Conclusion: Ideal for control synthesis by prescriptive games

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- Introduction
- Games from Dual Decomposition
- **Game Formulation for Distributed Disturbance Rejection**
- Dynamic Problems

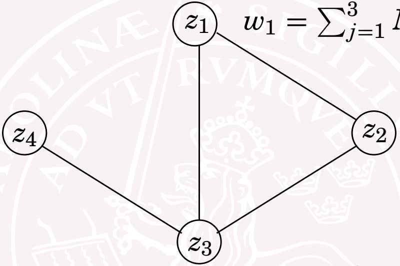
Disturbance Rejection with Graph Structure

$\ell_1(z_1) = |z_1|^2$

$w_1 = \sum_{j=1}^{3} M_{1j} z_j$

$\ell_4(z_4) = |z_4|^2$

$w_4 = \sum_{j=3}^{4} M_{4j} z_j$



$\ell_2(z_2) = |z_2|^2$

$w_2 = \sum_{j=1}^{3} M_{2j} z_j$

$\ell_3(z_3) = |z_3|^2$

$w_3 = \sum_{j=1}^{4} M_{3j} z_j$

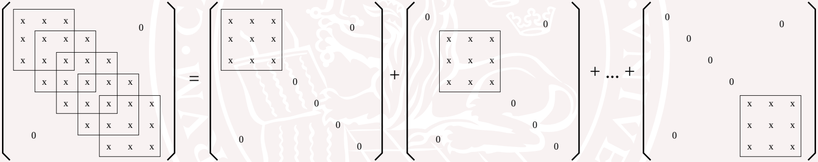
The disturbance rejection specification

$$\gamma^2 \geq \max_{|w|\leq 1} \sum_i |z_i|^2 \text{ subject to } \sum_j M_{ij} z_j = w_i$$

can be re-written as $M^T M \geq \gamma^{-2} I$. Is there a game to test this?

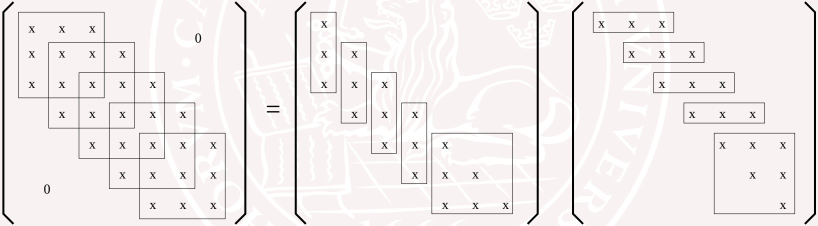
A Matrix Decomposition Theorem

The sparse matrix on the left is positive semi-definite if and only if it can be written as a sum of positive semi-definite matrices with the structure on the right.



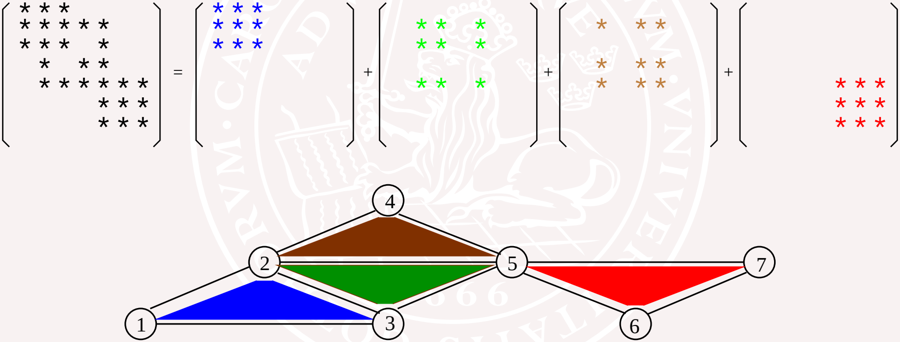
Proof idea

The decomposition follows immediately from the band structure of the Cholesky factors:



Generalization

Cholesky factors inherit the sparsity structure of the symmetric matrix if and only if the sparsity pattern corresponds to a "chordal" graph.



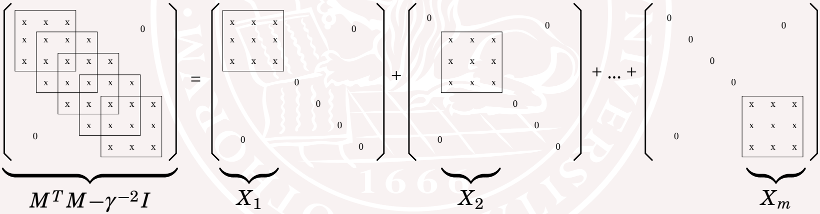
[Blair & Peyton, An introduction to chordal graphs and clique trees, 1992]

Distributed Disturbance Rejection as a Game

The disturbance rejection specification

$$\gamma^2 \geq \max_{|w|\leq 1} \sum_i |z_i|^2 \text{ subject to } \sum_j M_{ij} z_j = w_i$$

can be re-written as $M^T M \geq \gamma^{-2} I$. The condition



with $X_i \geq 0$ can be solved as a game using dual decomposition! However, the method works only for chordal graphs...

Outline

- Introduction
- Games from Dual Decomposition
- Distributed Disturbance Rejection as a Game
- **Dynamic Problems**

A control problem with graph structure

$$\begin{bmatrix} x_1(\tau+1) \\ x_2(\tau+1) \\ \vdots \\ x_{\mathcal{J}}(\tau+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & & 0 \\ A_{21} & \ddots & \ddots & \\ & \ddots & \ddots & A_{(\mathcal{J}-1)\mathcal{J}} \\ 0 & & A_{\mathcal{J}(\mathcal{J}-1)} & A_{\mathcal{J}\mathcal{J}} \end{bmatrix} \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \\ \vdots \\ x_{\mathcal{J}}(\tau) \end{bmatrix} + \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_{\mathcal{J}}(\tau) \end{bmatrix}$$

Minimize the convex objective $\sum_{t=0}^{N} \underbrace{\sum_{i=1}^{\mathcal{J}} \ell_i(x_i(\tau), u_i(\tau))}_{\ell(x(\tau),u(\tau))}$

with convex constraints $x_i(\tau) \in X_i$, $u_i(\tau) \in U_i$ and $x(0) = \bar{x}$.

Decomposing the problem

$$\text{Minimize } \sum_{t=0}^{N} \ell(x(\tau), u(\tau))$$

subject to

$$\begin{bmatrix} x_1(\tau+1) \\ x_2(\tau+1) \\ \vdots \\ x_J(\tau+1) \end{bmatrix} = \begin{bmatrix} A_{11}x_1(\tau) \\ A_{22}x_2(\tau) \\ \vdots \\ A_{JJ}x_J(\tau) \end{bmatrix} + \begin{bmatrix} v_1(\tau) \\ v_2(\tau) \\ \vdots \\ v_J(\tau) \end{bmatrix} + \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_J(\tau) \end{bmatrix}$$

where $x(0) = \bar{x}$ and

$$v_i = \sum_{j \neq i} A_{ij} x_j$$

holds for all i.

Decomposing the Cost Function

$$\max_p \min_{u,v,x} \sum_{\tau=0}^{N} \sum_{i=1}^{\mathcal{J}} \left[\ell_i(x_i,u_i) + p_i^T \left(v_i - \textstyle\sum_{j\neq i} A_{ij} x_j \right) \right]$$

$$= \max_p \sum_i \min_{u_i,x_i} \sum_{\tau=0}^{N} \underbrace{\left[\ell_i(x_i,u_i) + p_i^T v_i - x_i^T \left(\textstyle\sum_{j\neq i} A_{ji}^T p_j \right) \right]}_{\ell_i^p(x_i,u_i,v_i)}$$

so, given the sequences $\{p_j(t)\}_{t=0}^N$, agent i should minimize

$$\underbrace{\sum_{\tau=0}^{N} \ell_i(x_i,u_i)}_{\text{local cost}} + \overbrace{\sum_{\tau=0}^{N} p_i^T v_i}^{\text{what he expects others to charge him}} - \underbrace{\sum_{\tau=0}^{N} x_i^T \left(\textstyle\sum_{j\neq i} A_{ji}^T p_j \right)}_{\text{what he is payed by others}}$$

subject to $x_i(t+1) = A_{ii} x_i(t) + v_i(t) + u_i(t)$ and $x_i(0) = \bar{x}_i$.

Conclusions

- Convex sparse minimization with additive objective can be converted to game using dual decomposition
- Distributed disturbance rejection can be written as game, but so far only for chordal graphs