

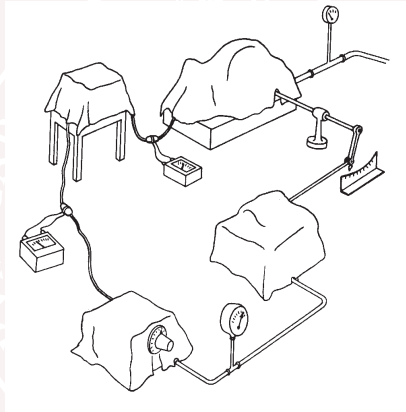


Distributed Verification of Sparse Systems

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Can Systems be Certified Distributively?



Componentwise performance verification without global model?

Outline

- Introduction
- **Distributed Positive Test for Matrices**
- Distributed Performance Verification of Linear Systems
- Open problems

A Matrix Decomposition Theorem

The sparse matrix on the left is positive semi-definite if and only if it can be written as a sum of positive semi-definite matrices with the structure on the right.

The diagram shows a large sparse matrix on the left, which is equal to a sum of several smaller matrices. The large matrix has a block-tridiagonal structure with 3x3 blocks on the main diagonal and 0s elsewhere. The sum consists of three terms: a matrix with a 3x3 block in the top-left corner and 0s elsewhere, followed by an ellipsis, and a matrix with a 3x3 block in the bottom-right corner and 0s elsewhere. Each 3x3 block contains 'x' characters representing non-zero entries.

$$\begin{pmatrix} \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & \\ & & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & \\ 0 & & & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & 0 \\ & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & \\ & & & 0 \\ 0 & & & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & & & 0 \\ & 0 & & 0 \\ & & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & \\ 0 & & & 0 \end{pmatrix}$$

Proof idea

The decomposition follows immediately from the band structure of the Cholesky factors:

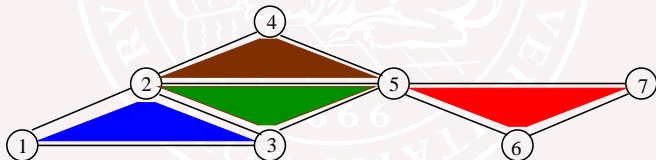
The diagram shows the decomposition of a sparse matrix into three factors, represented by their band structures. The matrix is shown as a product of three matrices, each enclosed in large parentheses and separated by an equals sign.

- Left Matrix:** A sparse matrix with a banded structure. The non-zero elements (marked with 'x') are arranged in a pattern that suggests a banded structure. A '0' is placed to the right of the matrix, and another '0' is placed below the matrix.
- Middle Matrix:** A sparse matrix with a banded structure. The non-zero elements are arranged in a pattern that suggests a banded structure.
- Right Matrix:** A sparse matrix with a banded structure. The non-zero elements are arranged in a pattern that suggests a banded structure.

Generalization

Cholesky factors inherit the sparsity structure of the symmetric matrix if and only if the sparsity pattern corresponds to a “chordal” graph.

$$\begin{pmatrix} * & * & * & & & & & \\ * & * & * & * & * & & & \\ * & * & * & & * & & & \\ * & & * & * & * & & & \\ * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * \\ & & * & * & * & * & * & * \\ & & & * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & & & & & \\ & * & * & * & & & & \\ & & * & * & * & & & \\ & & & * & * & * & & \\ & & & & * & * & * & \\ & & & & & * & * & * \end{pmatrix} + \begin{pmatrix} & & & * & * & * & & \\ & & & * & * & * & & \\ & & & * & * & * & & \\ & & & & * & * & * & \\ & & & & & * & * & * \end{pmatrix} + \begin{pmatrix} & & & & * & & * & * \\ & & & & * & & * & * \\ & & & & * & & * & * \\ & & & & & * & * & * \end{pmatrix} + \begin{pmatrix} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & & & & & \\ & * & * & * & & & & \\ & & * & * & * & & & \\ & & & * & * & * & & \\ & & & & * & * & * & \\ & & & & & * & * & * \end{pmatrix}$$



[Blair & Peyton, An introduction to chordal graphs and clique trees, 1992]

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A Sparse Stability Test

For the sparse matrix A , let the left hand side illustrate the structure of $(sI - A)^*(sI - A)$. Then the matrix is stable if and only if the right hand side split can be done with all squares positive definite for s in the right half plane.

$$(sI - A)^*(sI - A) = \begin{pmatrix} \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & \\ & & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & \\ 0 & & & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & 0 \\ & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & \boxed{\begin{matrix} x & x & x \\ x & x & x \\ x & x & x \end{matrix}} \end{pmatrix}$$

Hence global stability can always be verified by local tests!

A Sparse Passivity Test

Suppose

$$\begin{aligned} \dot{x} &= Ax + Bx + w & x(0) &= 0 \\ y &= Cx \end{aligned}$$

Then

$$\int_0^T \left(\gamma^2 u(t)y(t) + |w(t)|^2 \right) dt \geq 0 \quad \text{for all } u, w, T$$

if and only if the matrix

$$\begin{bmatrix} (sI - A)^*(sI - A) & \gamma^2 C^T - (sI - A)^* B \\ \gamma^2 C - B^*(sI - A) & B^T B \end{bmatrix}$$

is positive semi-definite for $\text{Re } s \geq 0$.

Passivity can be tested componentwise without conservatism!

Conclusions

- A “general” distributed non-conservative performance test
- Relation to existing sufficient criteria
- Storage function interpretation
- Non-linear versions?