Scalable Control of Convex-Monotone Systems

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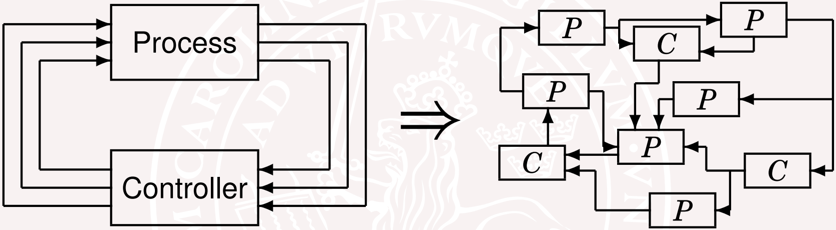
Towards a Scalable Control Theory



Can we find distributed controllers by distributed computation?

Outline

- **Positive and Convex-Monotone Systems**
- Voltage Stability
- HIV and Cancer Treatment

Positive systems

A linear system is called *positive* if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

$$\frac{dx}{dt} = Ax + Bu \qquad\qquad y = Cx$$

Equivalently, A, B and C have nonnegative coefficients except for the diagonal of A.

Examples:

- Probabilistic models.
- Economic systems.
- Chemical reactions.
- Traffic Networks.

Positive Systems and Nonnegative Matrices

Classics:

Mathematics: Perron (1907) and Frobenius (1912)

Economics: Leontief (1936)

Books:

Nonnegative matrices: Berman and Plemmons (1979)

Dynamical Systems: Luenberger (1979)

Recent control related work:

Biology inspired theory: Angeli and Sontag (2003)

Synthesis by linear programming: Rami and Tadeo (2007)

Switched systems: Liu (2009), Fornasini and Valcher (2010)

Distributed control: Tanaka and Langbort (2010)

Robust control: Briat (2013)

Stability of Positive systems

Suppose the matrix A has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exits a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

(iii) There exits a vector $z > 0$ such that $A^T z < 0$.

(iv) There is a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$

Lyapunov Functions of Positive systems

Solving the three alternative inequalities gives three different Lyapunov functions:

$A\xi < 0$ $\qquad$ $A^T P + PA \prec 0$ $\qquad$ $A^T z < 0$

$V(x) = \max_k (x_k/\xi_k)$ $\qquad$ $V(x) = x^T P x$ $\qquad$ $V(x) = z^T x$

A Scalable Stability Test for Positive Systems

Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

…

Verification is scalable!

A Distributed Search for Stabilizing Gains

Suppose $\begin{bmatrix} a_{11}-\ell_1 & a_{12} & 0 & a_{14} \\ a_{21}+\ell_1 & a_{22}-\ell_2 & a_{23} & 0 \\ 0 & a_{32}+\ell_2 & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \geq 0$ for $\ell_1, \ell_2 \in [0,1]$.

For stabilizing gains ℓ_1, ℓ_2, find $0 < \mu_k < \xi_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).

Examples: Transportation Networks

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics

Externally Positive Systems

$\mathbf{G} \in \mathbb{RH}_\infty^{m\times n}$ is called *externally positive* if if the corresponding impulse response $g(t)$ is nonnegative for all t. The set of all such matrices is denoted $\mathbb{PH}_\infty^{m\times n}$.

Suppose $\mathbf{G}, \mathbf{H} \in \mathbb{PH}_\infty^{n\times n}$. Then

- $\mathbf{GH} \in \mathbb{PH}_\infty^{n\times n}$
- $a\mathbf{G} + b\mathbf{H} \in \mathbb{PH}_\infty^{n\times n}$ when $a, b \in \mathbb{R}_+$.
- $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.
- $(I - \mathbf{G})^{-1} \in \mathbb{PH}_\infty^{n\times n}$ if and only if $\mathbf{G}(0)$ is Schur.

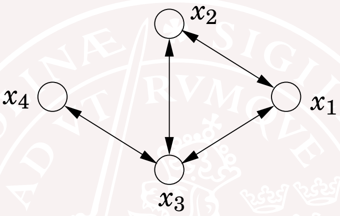
Positively Dominated Systems

$\mathbf{G} \in \mathbb{RH}_\infty^{m\times n}$ is called *positively dominated* if $|\mathbf{G}_{jk}(i\omega)| \le \mathbf{G}_{jk}(0)$ for $\omega \in \mathbb{R}$. The set of all such matrices is denoted $\mathbb{DH}_\infty^{m\times n}$.

Suppose $\mathbf{G}, \mathbf{H} \in \mathbb{DH}_\infty^{n\times n}$. Then

- $\mathbf{GH} \in \mathbb{DH}_\infty^{n\times n}$
- $a\mathbf{G} + b\mathbf{H} \in \mathbb{DH}_\infty^{n\times n}$ when $a, b \in \mathbb{R}_+$.
- $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.
- $(I - \mathbf{G})^{-1} \in \mathbb{DH}_\infty^{n\times n}$ if and only if $\mathbf{G}(0)$ is Schur.

Example 3: Mass-spring system



$$\ddot{x}_i + d_i\dot{x} + k_i x_i = \sum_j \ell_{ij}(x_j - x_i) + w_i$$

$$\left(s^2 + d_i s + k_i + \sum_j \bar{\ell}_{ij}\right) X_i(s) = \sum_j \left(\ell_{ij} X_j(s) + (\bar{\ell}_{ij} - \ell_{ij}) X_i(s)\right) + W_i(s)$$

$$X = (\mathbf{A} + \mathbf{E}L\mathbf{F})X + \mathbf{B}W$$

The transfer matrices $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{A} + \mathbf{E}L\mathbf{F}$ are positively dominated for all $L \in \mathcal{D}$ provided that $d_i \geq k_i + \sum_j \bar{\ell}_{ij}$.

Max-separable Lyapunov Functions

Max-separable: $V(x) = \max\{V_1(x_1), \ldots, V_n(x_n)\}$

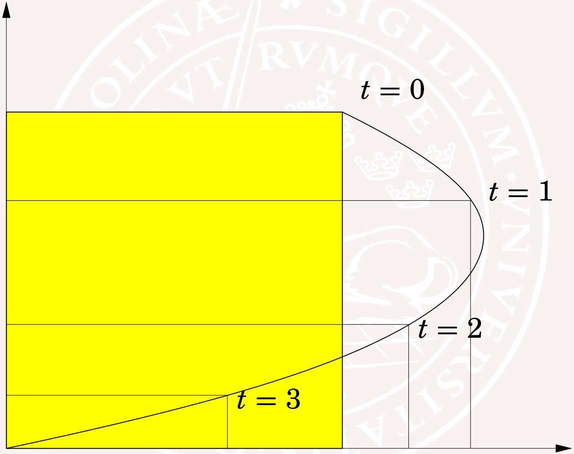
Theorem. Let $\dot{x} = f(x)$ be a monotone system such that the origin globally asymptotically stable and the compact set $X \subset \mathbb{R}^n_+$ is invariant. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1, \ldots, n$, such that $V(x) = \max\{V_1(x_1), \ldots, V_n(x_n)\}$ satisfies

$$\frac{d}{dt} V(x(t)) = -V(x(t))$$

along all trajectories in X.

[Rantzer, Rüffer, Dirr, CDC-13]

Proof idea



Convex-Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = a$$

is a *monotone system* if its linearization is a positive system. It is a *convex monotone system* if every row of f is also convex.

Theorem. [Rantzer/ Bernhardsson (2014)]

For a convex monotone system $\dot{x} = f(x, u)$, each component of the trajectory $\phi_t(a, u)$ is a convex function of (a, u).

Outline

- ◦ Positive and Convex -Monotone Systems
- • **Voltage Stability**
- ◦ HIV and Cancer Treatment

One Transmission Line

The power $p = iu_2$ delivered to the load is upper bounded by

$$p = i(u_1 - Ri) \leq \frac{u_1^2}{4R}.$$

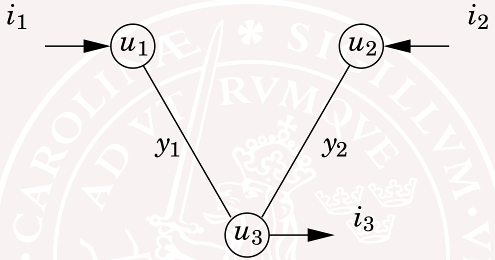
An active load:

$$\frac{di}{dt} = \frac{\widehat{p}}{u_1 - Ri} - i.$$

where $\widehat{p}$ is the power demand.

Voltage collapse occurs if $\widehat{p}$ is too large!

Two Transmission Lines



Node 3 is an active load with

$$\frac{di_3}{dt} = \frac{\widehat{p}(y_1 + y_2)}{y_1 u_1 + y_2 u_2 - i_3} - i_3$$

For constant generator voltages u_1 and u_2, the load voltage $u_3 = y_1 u_1 + y_2 u_2 - i_3$ could shrink to zero in finite time, which means voltage collapse.

Arbitrary Networks

Voltages at generators u^G and loads u^L are mapped into external currents i^G and i^L according to

$$\begin{bmatrix} -i^G(t) \\ i^L(t) \end{bmatrix} = \begin{bmatrix} Y^{GG} & Y^{GL} \\ Y^{LG} & Y^{LL} \end{bmatrix} \begin{bmatrix} u^G(t) \\ u^L(t) \end{bmatrix}$$

The load model: $\frac{di_k^L}{dt}(t) = \frac{\widehat{p}_k}{u_k^L(t)} - i_k^L(t)$ gives

$$\frac{di^L}{dt}(t) = \widehat{p}./[(Y^{LL})^{-1}(i^L - Y^{LG}u^G)] - i^L(t)$$

This system is convex-monotone with state i^L and input $-u^G$, so

$$i^G, -u^L, i^L, \frac{di^L}{dt} \text{ and } \frac{di^G}{dt}$$

are all convex functions of u^G

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Combination Therapy is a Control Problem

Evolutionary dynamics:

$$\dot{x} = \left(A - \sum_i u_i D^i \right) x$$

Each state x_k is the concentration of a mutant. (There can be hundreds!) Each input u_i is a drug dosage.

A describes the mutation dynamics without drugs, while $D^1, \ldots, D^m$ are diagonal matrices modeling drug effects.

Determine $u_1, \ldots, u_m \geq 0$ with $u_1 + \cdots + u_m \leq 1$ such that x decays as fast as possible!

[Jonsson, Rantzer,Murray, ACC 2014]

Optimizing Decay Rate

Stability of the matrix $A - \sum_i u_i D^i + \gamma I$ is equivalent to existence of $\xi > 0$ with

$$(A - \sum_i u_i D^i + \gamma I)\xi < 0$$

For row k, this means

$$A_k \xi - \sum_i u_i D^i_k \xi_k + \gamma \xi_k < 0$$

or equivalently

$$\frac{A_k \xi}{\xi_k} - \sum_i u_i D^i_k + \gamma < 0$$

Maximizing γ is convex optimization in $(\log \xi_i, u_i, \gamma)$!

Using Measurements of Virus Concentrations

Evolutionary dynamics:

$$\dot{x}(t) = \left(A - \sum_i u_i(t) D^i \right) x(t)$$

Can we get faster decay using time-varying $u(t)$ based on measurements of $x(t)$?

Using Measurements of Virus Concentrations

The evolutionary dynamics can be written as a convex monotone system:

$$\frac{d}{dt} \log x_k(t) = \frac{A_k x(t)}{x_k(t)} - \sum_i u_i(t) D_k^i$$

Hence the decay of $\log x_k$ is a convex function of the input and optimal trajectories can be found even for large systems.

Example

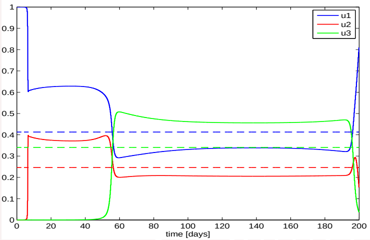
$$A = \begin{bmatrix} -\delta & \mu & \mu & 0 \\ \mu & -\delta & 0 & \mu \\ \mu & 0 & -\delta & \mu \\ 0 & \mu & \mu & -\delta \end{bmatrix}$$

clearance rate $\delta = 0.24\,\mathrm{day}^{-1}$, mutation rate $\mu = 10^{-4}\,\mathrm{day}^{-1}$ and replication rates for viral variants and therapies as follows

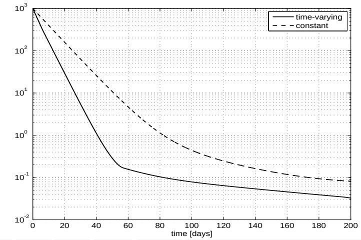
Virus variant	Therapy 1	Therapy 2	Therapy 3
Type 1 (x_1)	$D_1^1 = 0.05$	$D_1^2 = 0.10$	$D_1^3 = 0.30$
Type 2 (x_2)	$D_2^1 = 0.25$	$D_2^2 = 0.05$	$D_2^3 = 0.30$
Type 3 (x_3)	$D_3^1 = 0.10$	$D_3^2 = 0.30$	$D_3^3 = 0.30$
Type 4 (x_4)	$D_4^1 = 0.30$	$D_4^2 = 0.30$	$D_4^3 = 0.15$

Example

Optimized drug doses:



Total virus population:



Summary

- Scalability for Positive and Convex-Monotone Systems
- Voltage Stability
- HIV and Cancer Treatment