Controllability metrics, limitations and algorithms for complex networks

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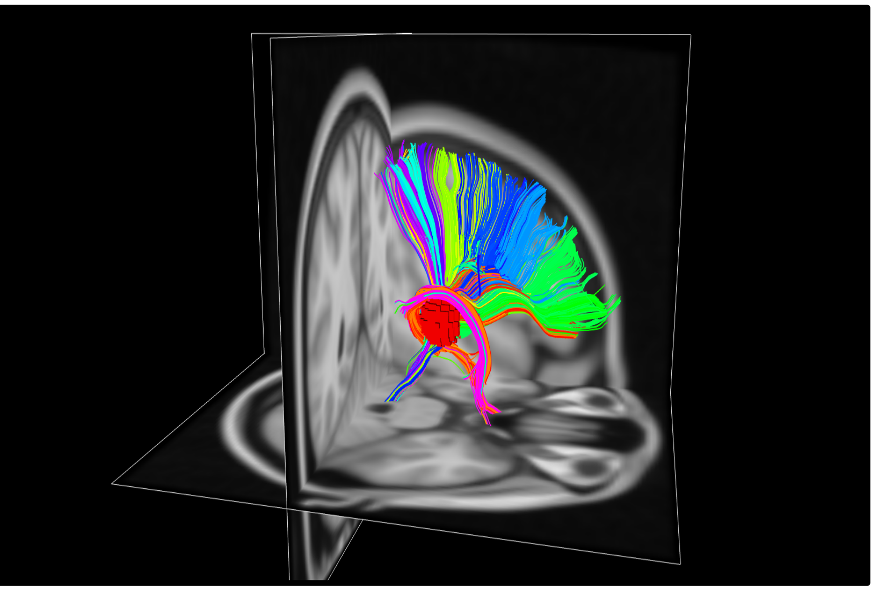
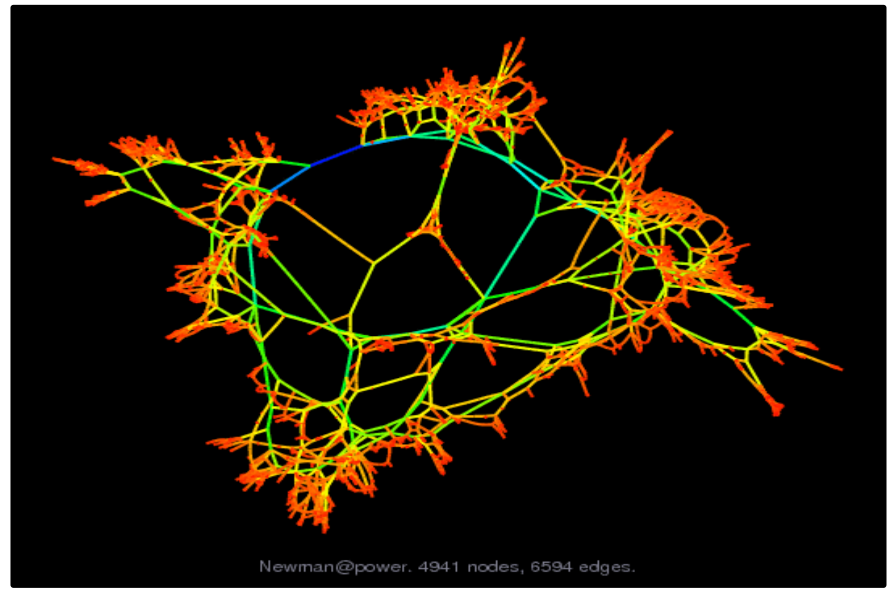
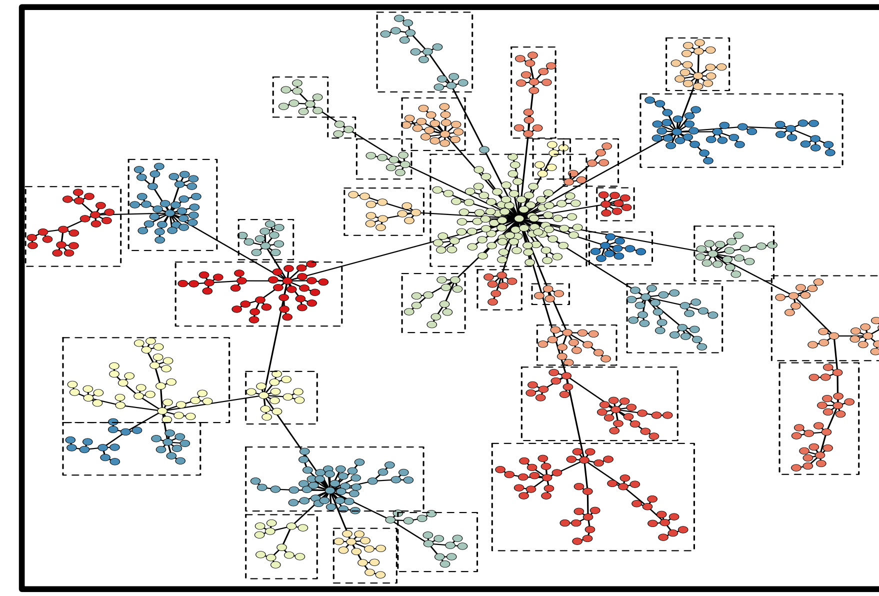
Growing Number of Applications



Brain network



Power network



Social network

✓ Y. Y. Liu, J. J. Slotine, and A. L. Barabasi, "Controllability of complex networks," Nature 2011.

✓ N. J. Cowan, E. J. Chastain, D. A. Vilhena, J. S. Freudenberg, and C. T. Bergstrom, "Nodal dynamics, not degree distributions, determine the structural controllability of complex networks," PLOS ONE, 2012.

✓ G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and B. Li, "Controlling complex networks: How much energy is needed?" Physical Review Letters, 2012.

✓ J. Sun and A. E. Motter, "Controllability transition and nonlocality in network control," Physical Review Letters, 2013.

✓ F. L. Cortesi, T. H. Summers, and J. Lygeros, "Submodularity of Energy Related Controllability Metrics," Arxiv preprint, 2014.

✓ A. Olshevsky, "Minimal Controllability Problems," Arxiv preprint, 2014.

Problem formulation

Consider a linear system

$$x(t+1) = Ax(t) + Bu(t)$$

where A is sparse $n \times n$ matrix (the interactions between the states is described by a graph) and

$$B = \begin{bmatrix} e_{i_1} & \cdots & e_{i_m} \end{bmatrix}$$

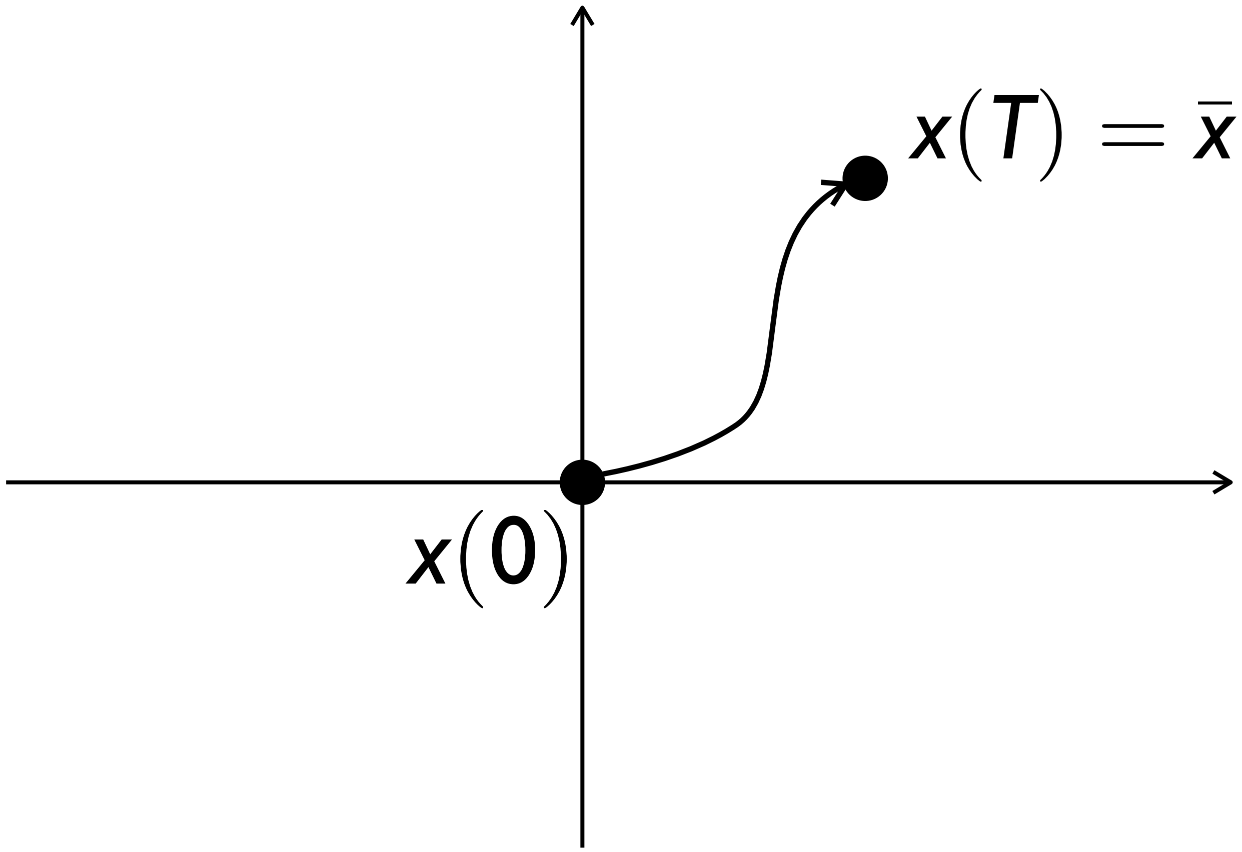
where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

Problem formulation

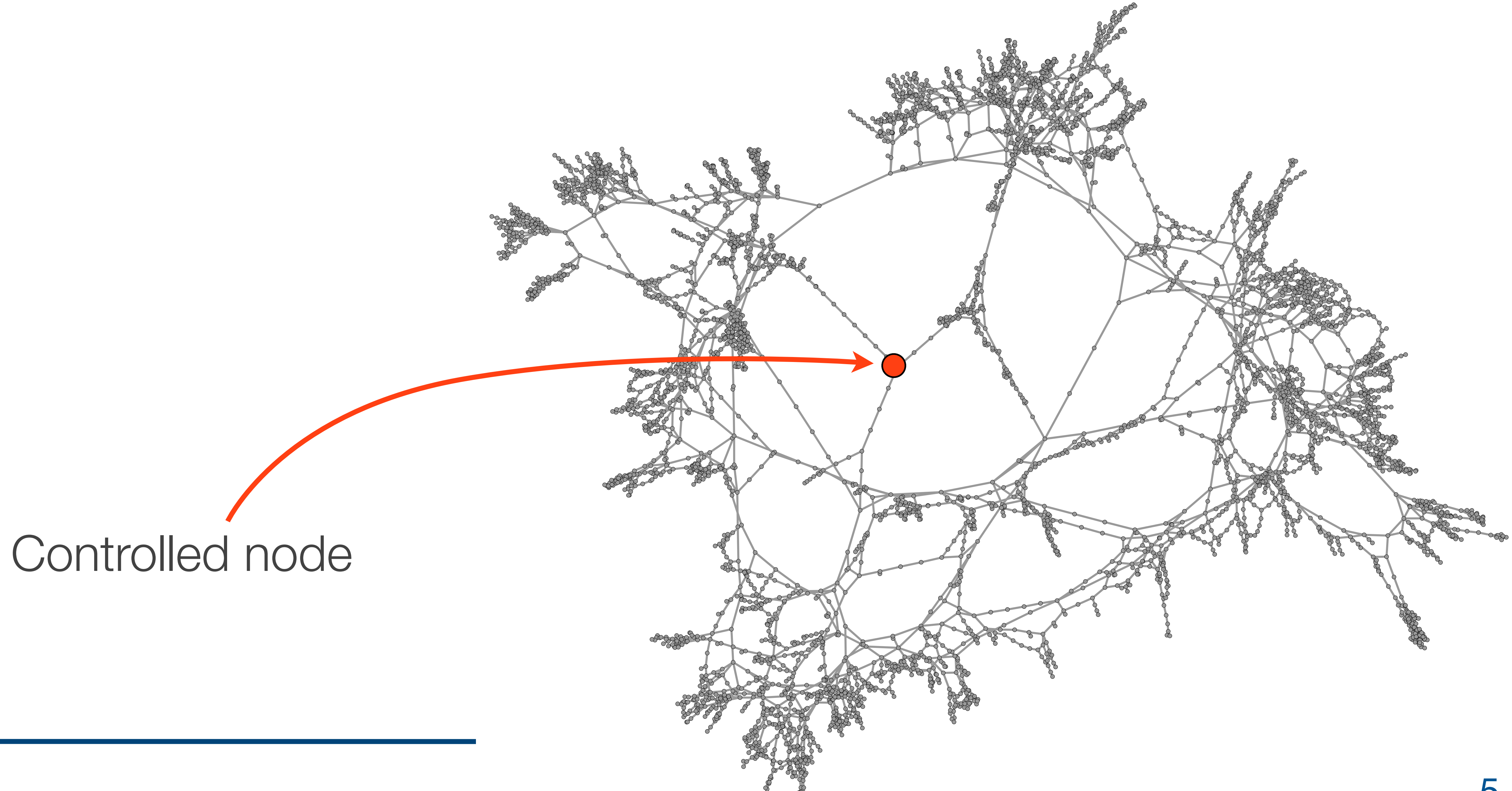
$$x(t+1) = Ax(t) + Bu(t)$$

CONTROLLABILTY is the possibility of streering the state from the initial state $x(0) = 0$ to an arbitrary final state $x(T) = \bar{x}$ by applying a suitable input sequence $u(0), u(1), \ldots, u(T-1)$.



Need of a controllability metric

Assume that the graph is strongly connected and that it has all the self-loops. Then to have that the resulting system is controllable (generically in the non-zero entries of A) it is enough that only one state is controlled.



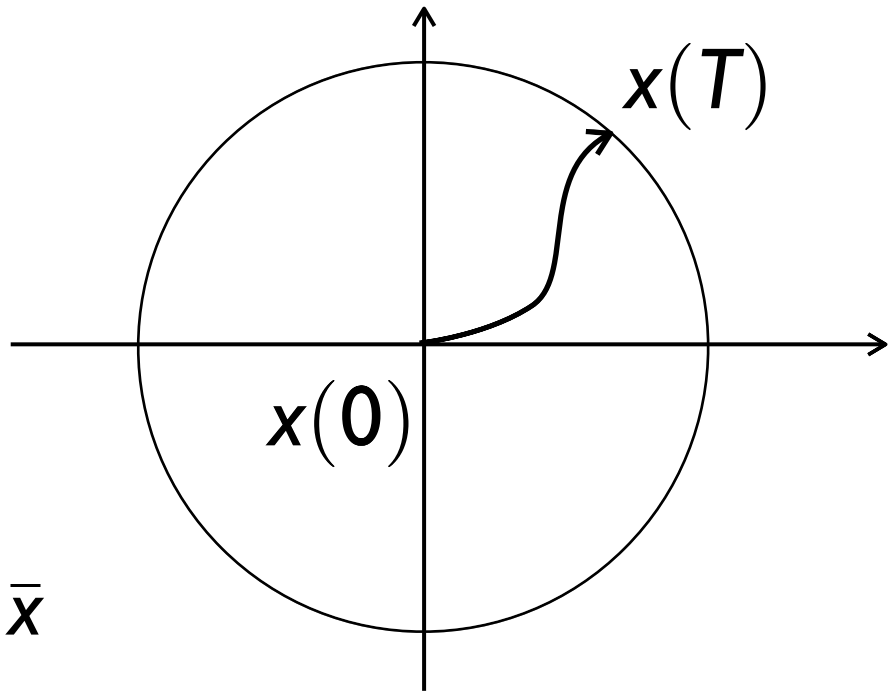
Need of a controllability metric

QUESTION: How controllable is the resulting system?

A controllability metric

One metric for describing the controllability degree is given by the controllability Gramian

$$W_T := \sum_{t=0}^{T-1} A^t B B^T (A^T)^t$$



Interpretation: For driving

$$x(0) = 0 \quad \longrightarrow \quad x(T) = \bar{x}$$

where $||\bar{x}||_2 = 1$, we need an input $u(0), u(1), \ldots, u(T-1)$ with L^2 energy $\bar{x}^T W_T^{-1} \bar{x}$.

The energy to drive the state from zero to a norm one state (in the worst case) is given by

$$Energy = \frac{1}{\lambda_{min}(W_T)}$$

A controllability metric

$$W_T := \sum_{t=0}^{T-1} A^t B B^T (A^T)^t$$ Controllability Gramian

$$Energy = \frac{1}{\lambda_{min}(W_T)}$$

Small $\lambda_{min}(W_T)$ ⟷ Small controllability degree

Large $\lambda_{min}(W_T)$ ⟷ Large controllability degree

A controllability metric

Alternative controllability metrics

$$W_T := \sum_{t=0}^{T-1} A^t B B^T (A^T)^t$$

$$\lambda_{min}(W_T)$$

$$\frac{1}{n}\mathrm{tr}(W_T^{-1})$$

$$\frac{1}{n}\ln\det(W_T)$$

$$\frac{1}{n}\mathrm{tr}(W_T)$$

THEOREM

Assume the matrix A symmetric. Fix any constant $0 < \mu < 1$ and let

$$n(\mu) := |\{\lambda \in \lambda(A) \ : \ |\lambda|^2 \leq \mu\}|$$

Then

$$\lambda_{min}(W_T) \leq \frac{1}{\mu(1-\mu)} \mu^{n(\mu)/m}$$

Few Nodes Cannot Control Symmetric Complex Networks

Let

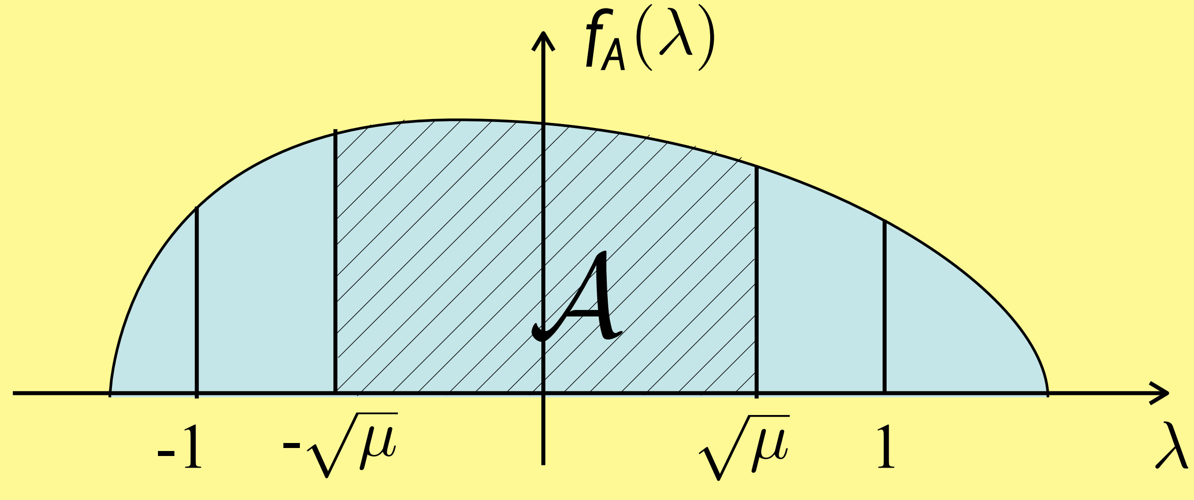
$$F_A(\bar{\lambda}) := \frac{|\{\lambda \in \lambda(A) \ : \ \lambda \leq \bar{\lambda}\}|}{n}$$

and

$$f_A(\bar{\lambda}) := \frac{dF(\bar{\lambda})}{d\bar{\lambda}}$$

Then

Distribution of eigenvalues



$$n(\mu) := |\{\lambda \in \lambda(A) \ : \ |\lambda|^2 \leq \mu\}| = n \int_{-\sqrt{\mu}}^{\sqrt{\mu}} f_A(\lambda) d\lambda = \mathcal{A}\, n$$

Hence the number $n(\mu)$ typically grows linearly in the network cardinality n.

$$\lambda_{min}(W_T) \leq \frac{1}{\mu(1-\mu)} \mu^{\mathcal{A}\frac{n}{m}}$$

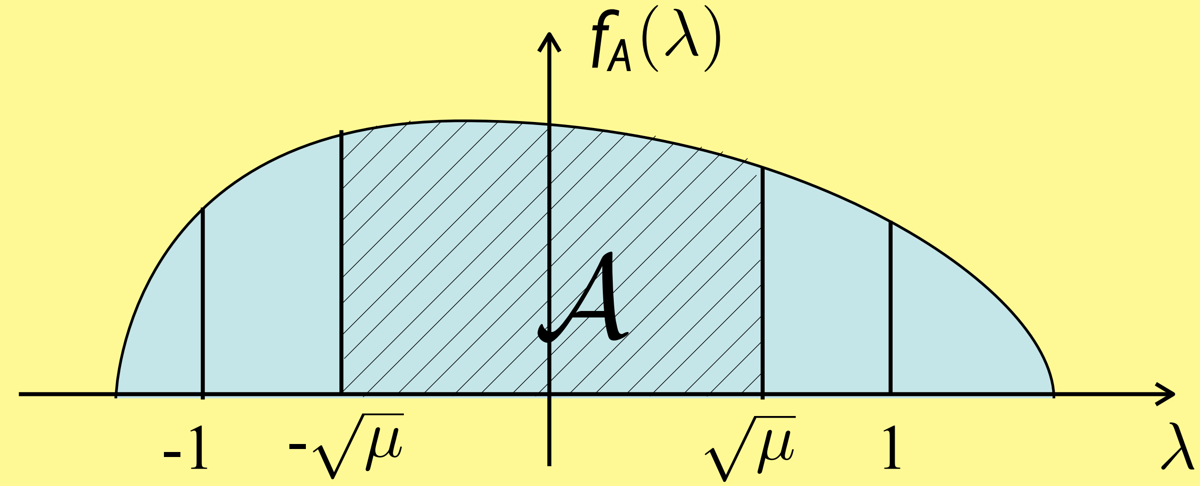
$\mu < 1$

Few Nodes Cannot Control Symmetric Complex Networks

$$\mu < 1$$

$$\lambda_{min}(W_T) \leq \frac{1}{\mu(1-\mu)} \mu^{\mathcal{A} \frac{n}{m}}$$

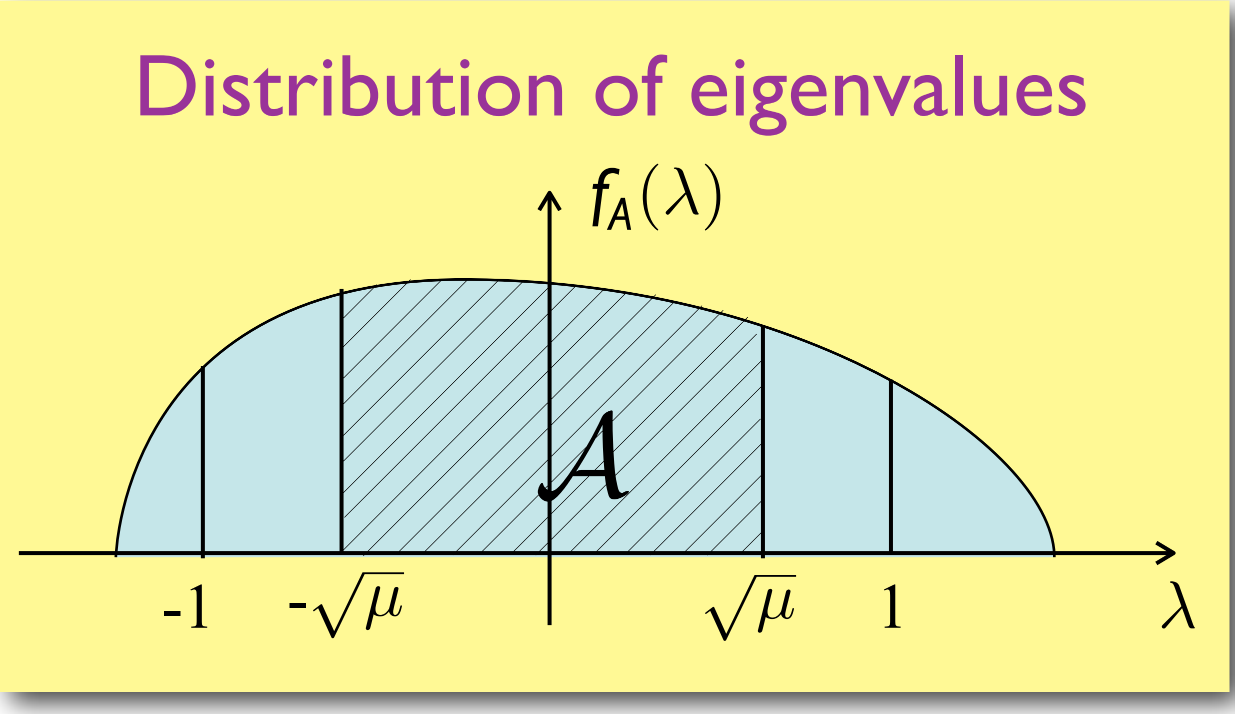
Distribution of eigenvalues



Few Nodes Cannot Control Symmetric Complex Networks

$\mu < 1$

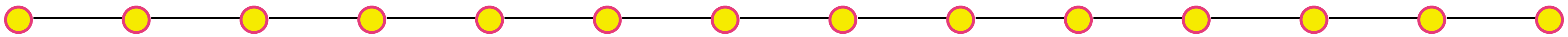
$$\lambda_{min}(W_T) \leq \frac{1}{\mu(1-\mu)} \mu^{\mathcal{A} \frac{n}{m}}$$



✓ For fixed number **m** of control nodes, the controllability degree **decreases exponentially** in the network cardinality **n**.

✓ To have a fixed controllability degree, number **m** of control nodes must grow **linearly** in the network cardinality **n**.

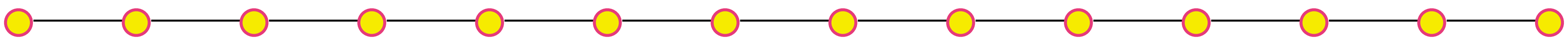
Example: symmetric line graph



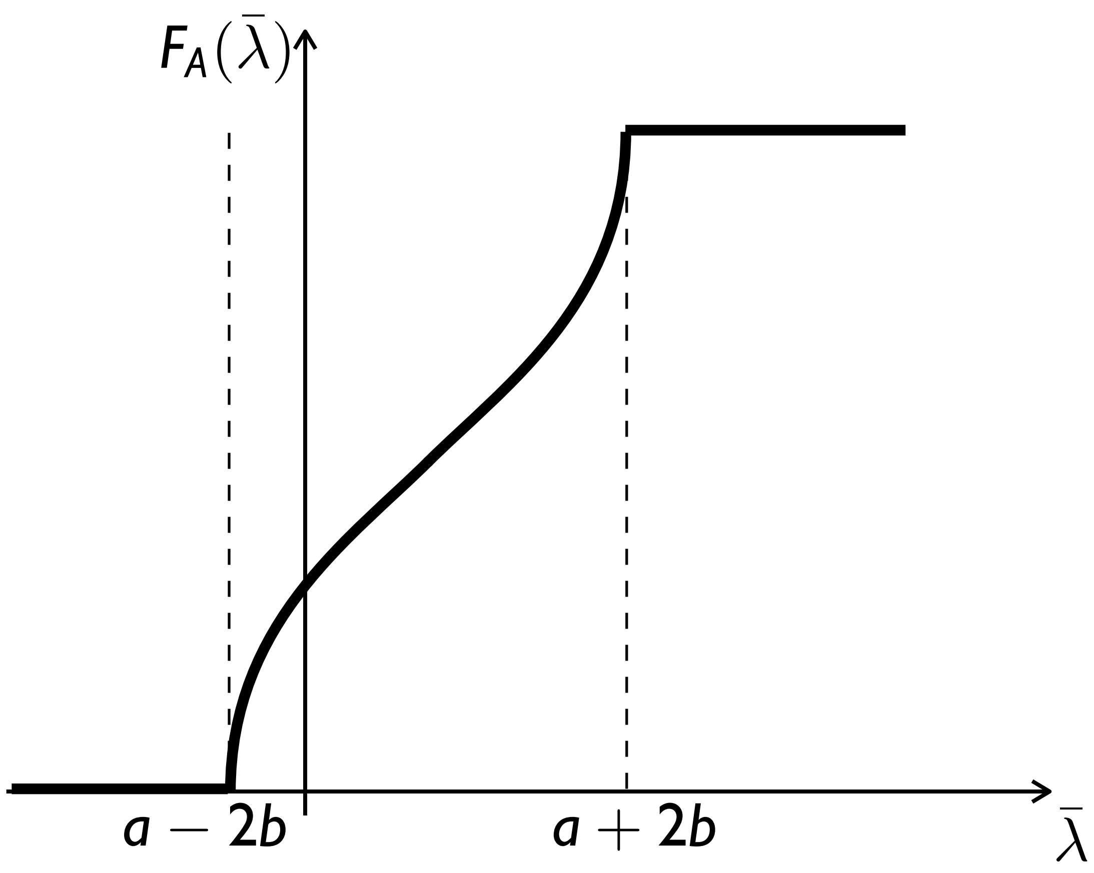
$$A = \begin{bmatrix} a & b & 0 & \cdots & \cdots & 0 & 0 \\ b & a & b & \cdots & \cdots & 0 & 0 \\ 0 & b & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a & b \\ 0 & 0 & 0 & \cdots & \cdots & b & a \end{bmatrix}$$

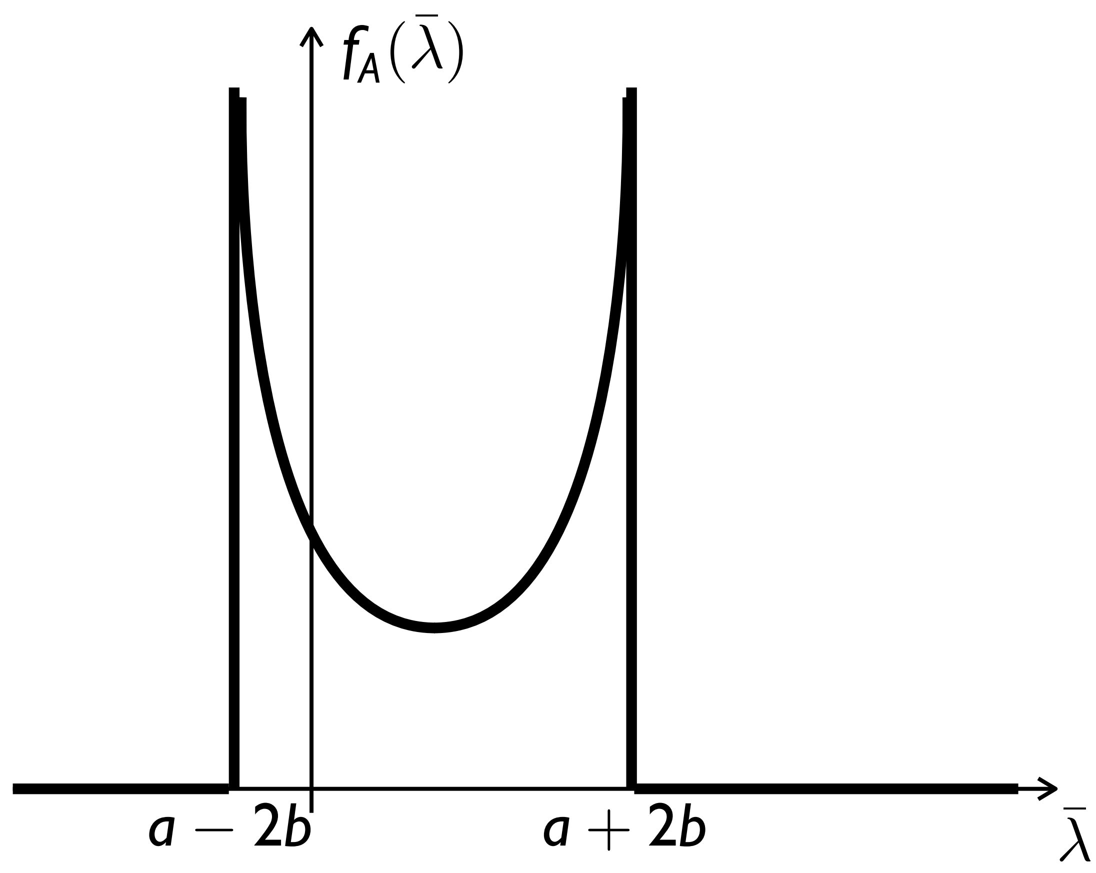
$$\lambda(A) = \left\{ a + 2b \cos\left(\frac{\pi}{n+1} k\right) : k = 1, 2, \ldots, n \right\}$$

Example: symmetric line graph

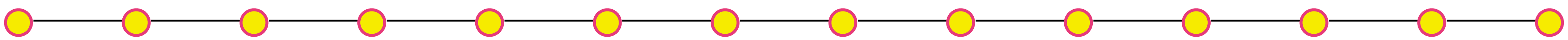


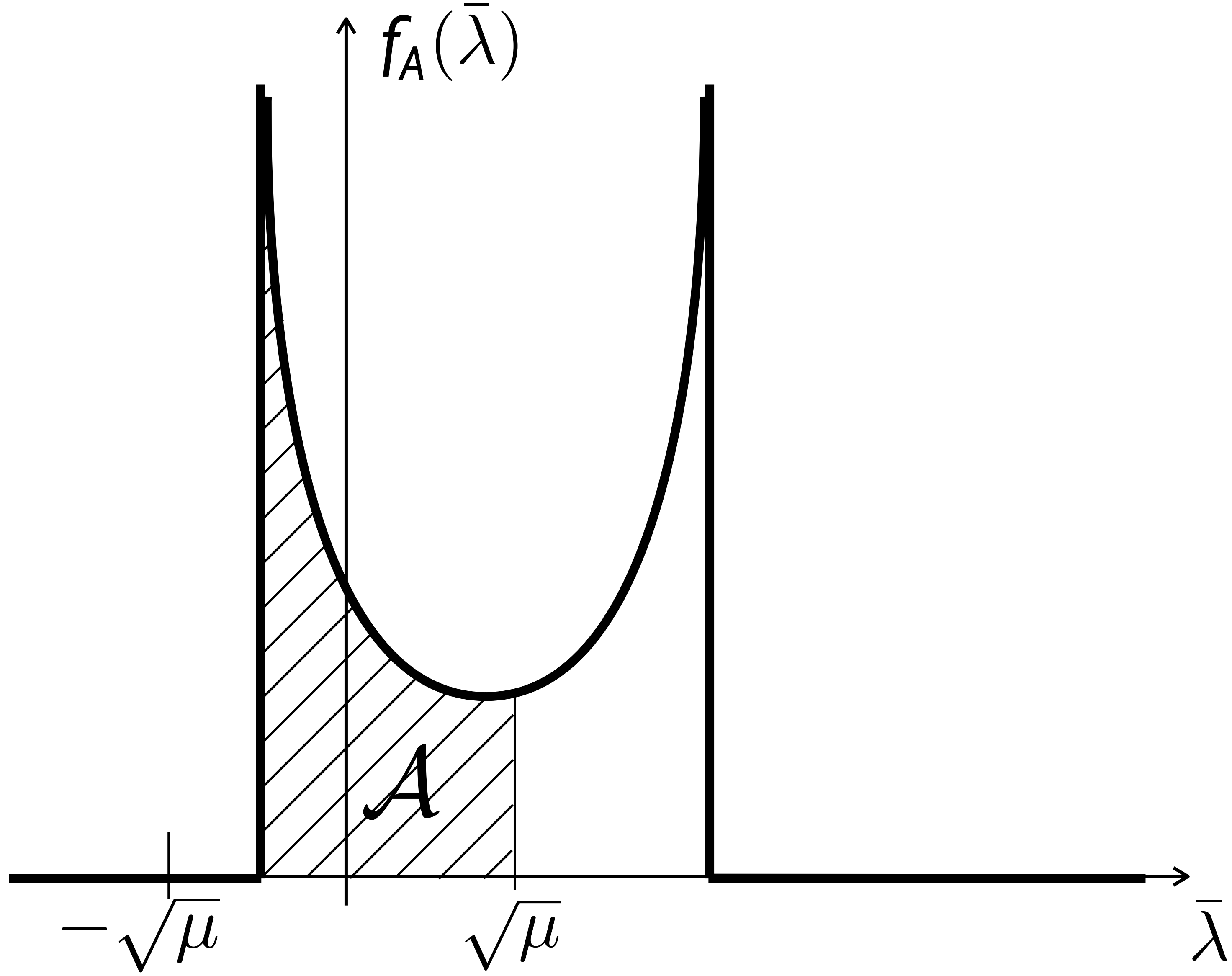
$$F_A(\bar{\lambda}) = 1 - \frac{1}{\pi}\arccos\left(\frac{\bar{\lambda} - a}{2b}\right) \qquad f_A(\bar{\lambda}) = \frac{1/\pi}{\sqrt{1 - \left(\frac{\bar{\lambda}-a}{2b}\right)^2}}$$





Example: symmetric line graph





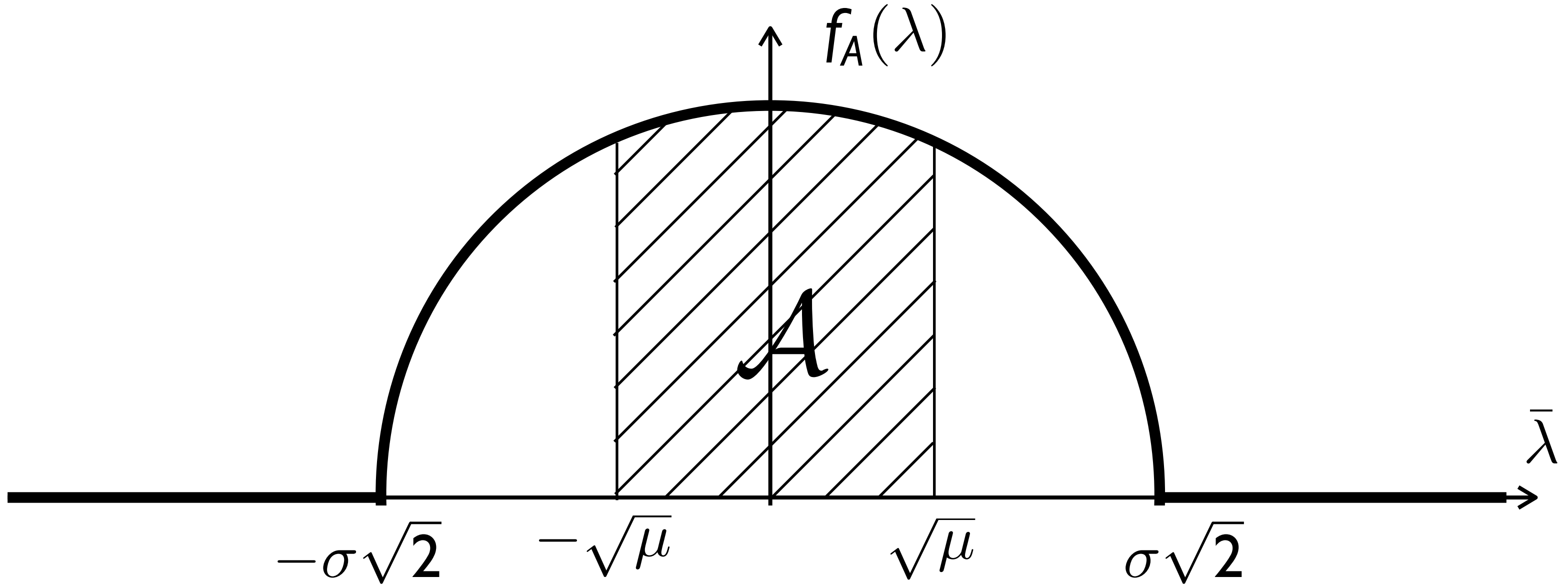
Example: Symmetric random matrix

Assume A is a symmetric matrix with a_{ij} iid random variables $\mathbb{E}[a_{ij}] = 0$ and $\mathbb{E}[a_{ij}^2] = \sigma^2/\sqrt{n}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Then (Wigner's semi-circle law)

$$f_A(\bar{\lambda}) \longrightarrow W(\bar{\lambda})$$



Asymmetric Complex Networks

For asymmetric networks the situation is more complex

- For “**isotropic**” networks (networks with no preferential directions) it seems that the situation is the same as for symmetric networks, namely they are difficult to control.

- For “**anisotropic**” networks (networks with a preferential direction) it seems that few nodes can indeed control large scale networks.

THEOREM

Assume the matrix A diagonalizable and let V an eigenvector matrix. Fix any constant $0 < \mu < 1$ and let

$$n(\mu) := |\{\lambda \in \lambda(A) \ : \ |\lambda|^2 \le \mu\}|$$

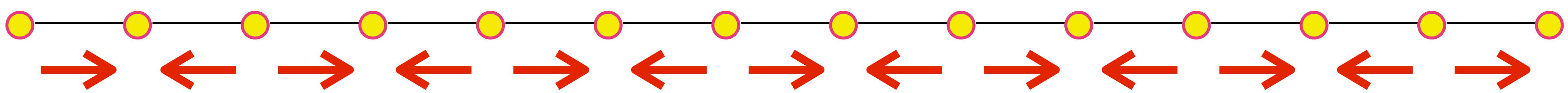
Then

$$\lambda_{min}(W_T) \le \mathrm{cond}(V)^2 \frac{1}{\mu(1-\mu)} \mu^{n(\mu)/m}$$

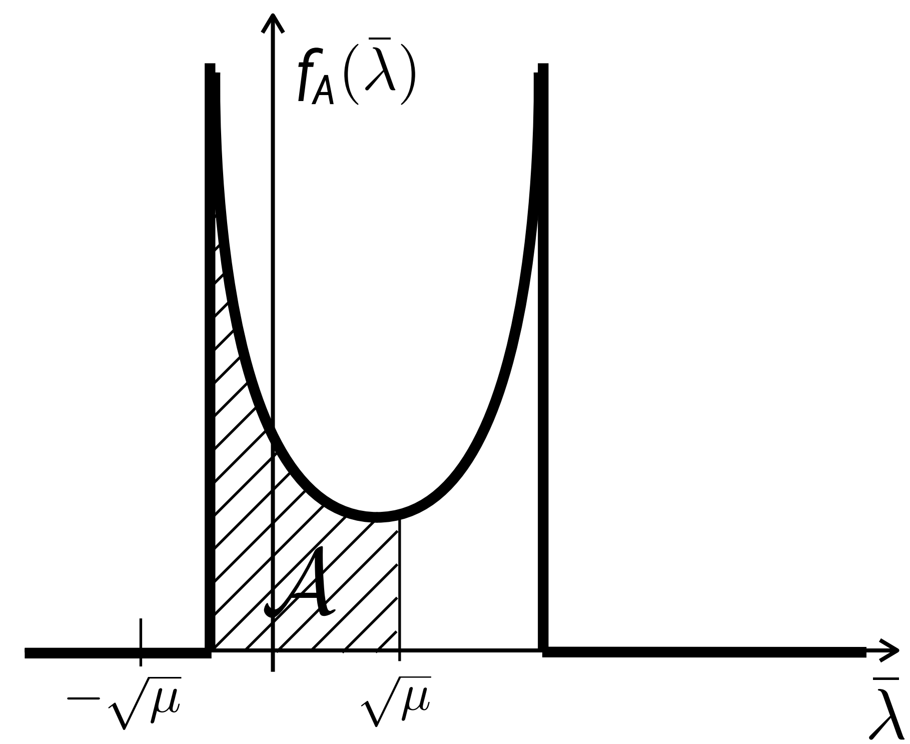
where $\mathrm{cond}(V) = \sigma_{\max}(V)/\sigma_{\min}(V)$ is the condition number of V.

Consequence: If the condition number stays bounded in the network dimension, than the network remains difficult to control.

Example: Asymmetric isotropic line graph



$$A := \frac{1}{3}\begin{bmatrix} 1 & 1/2 & 0 & 0 & \cdots \\ 2 & 1 & 2 & 0 & \cdots \\ 0 & 1/2 & 1 & 1/2 & \cdots \\ 0 & 0 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

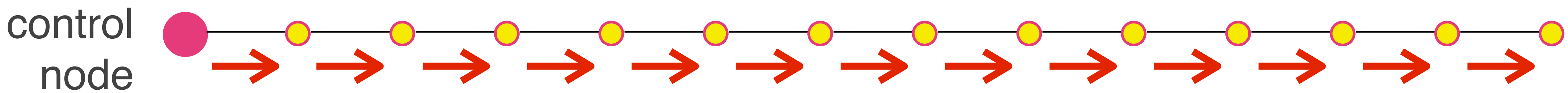


It can be found an eigenvector matrix V such that $\mathrm{cond}(V) = 2$.

$$\lambda(A) = \left\{ \frac{1}{3} + \frac{2}{3}\cos\frac{k\pi}{n+1}, \qquad k = 1, \ldots, n \right\}$$

This network is difficult to control as a symmetric network.

Example: Asymmetric anisotropic line graph



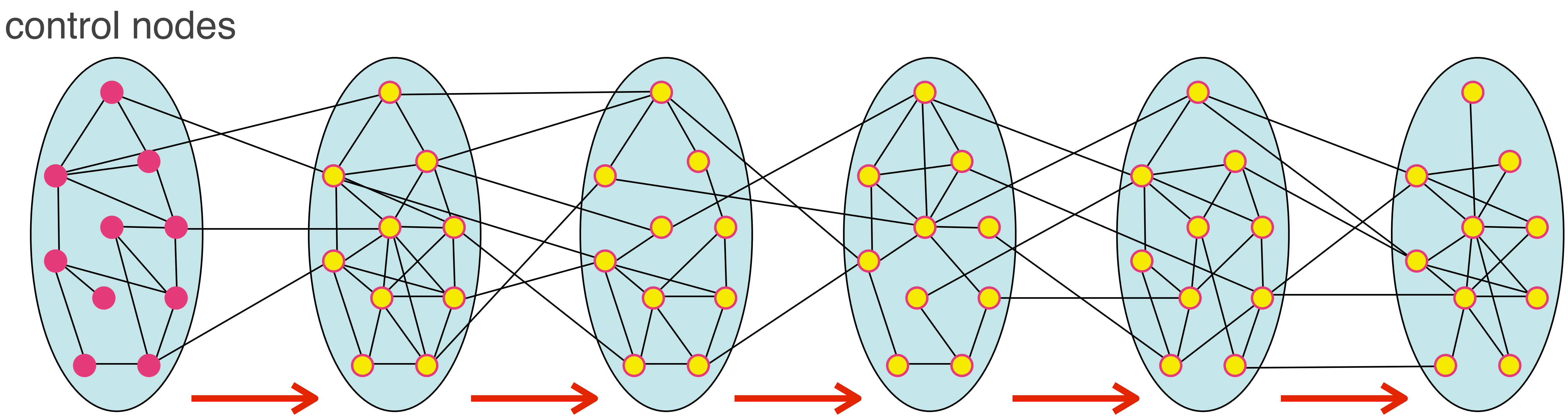
$$A = \begin{bmatrix} a & b & 0 & \cdots & \cdots & 0 & 0 \\ c & a & b & \cdots & \cdots & 0 & 0 \\ 0 & c & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a & b \\ 0 & 0 & 0 & \cdots & \cdots & c & a \end{bmatrix}$$

$$\lambda(A) = \left\{ a + 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}k\right) : k = 1, 2, \ldots, n \right\}$$

Given a, if c is sufficiently larger than b, than this network can be controlled with finite energy by the node on the extreme regardless the network dimension.

Exploiting spatial instability

Extension to more general graphs: controllable graphs



$$A = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & \cdots & 0 & 0 \\ C_1 & A_2 & B_2 & \cdots & \cdots & 0 & 0 \\ 0 & C_2 & A_3 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & A_{\ell-1} & B_{\ell-1} \\ 0 & 0 & 0 & \cdots & \cdots & C_{\ell-1} & A_\ell \end{bmatrix}$$

Extension to more general graphs

$$A = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & \cdots & 0 & 0 \\ C_1 & A_2 & B_2 & \cdots & \cdots & 0 & 0 \\ 0 & C_2 & A_3 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & A_{\ell-1} & B_{\ell-1} \\ 0 & 0 & 0 & \cdots & \cdots & C_{\ell-1} & A_\ell \end{bmatrix}$$

Given the matrices A_i, if the matrices C_i are sufficiently larger than the matrices B_i, than this network can be controlled with finite energy by the nodes on the extreme subgraph regardless the network dimension.

Extension to more general graphs: uncontrollable graphs

Assume A is a stochastic matrix (diffusion dynamics, consensus, ...). This means that

$$A1 = 1$$

where 1 is the vector with entries equal to one. Let w be a vector such that

$$w^T A = w^T \qquad w^T 1 = 1$$

which means that w is the invariant measure of the Markov chain associated with A. It is known that the entries of w represent the nodes "centrality" in the network (the bigger the more important).

Result: If the entries of w are all $\leq \frac{cost}{n}$ (all nodes have similar centrality) then the associated network is difficult to control.

In the symmetric case the entries of w are $1/n$.

Decoupled control strategy: "Divide et Impera"

Network partitioning Partition $\mathcal{V} = \{1, \dots, n\}$ into N disjoint sets $\mathcal{V}_1, \dots, \mathcal{V}_N$. After relabeling of states and inputs, the matrices read as

$$A = \begin{bmatrix} A_1 & \cdots & A_{1N} \\ \vdots & \vdots & \vdots \\ A_{N1} & \cdots & A_N \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_N \end{bmatrix},$$

The networks dynamics can be written as the interconnection of N subsystems of the form

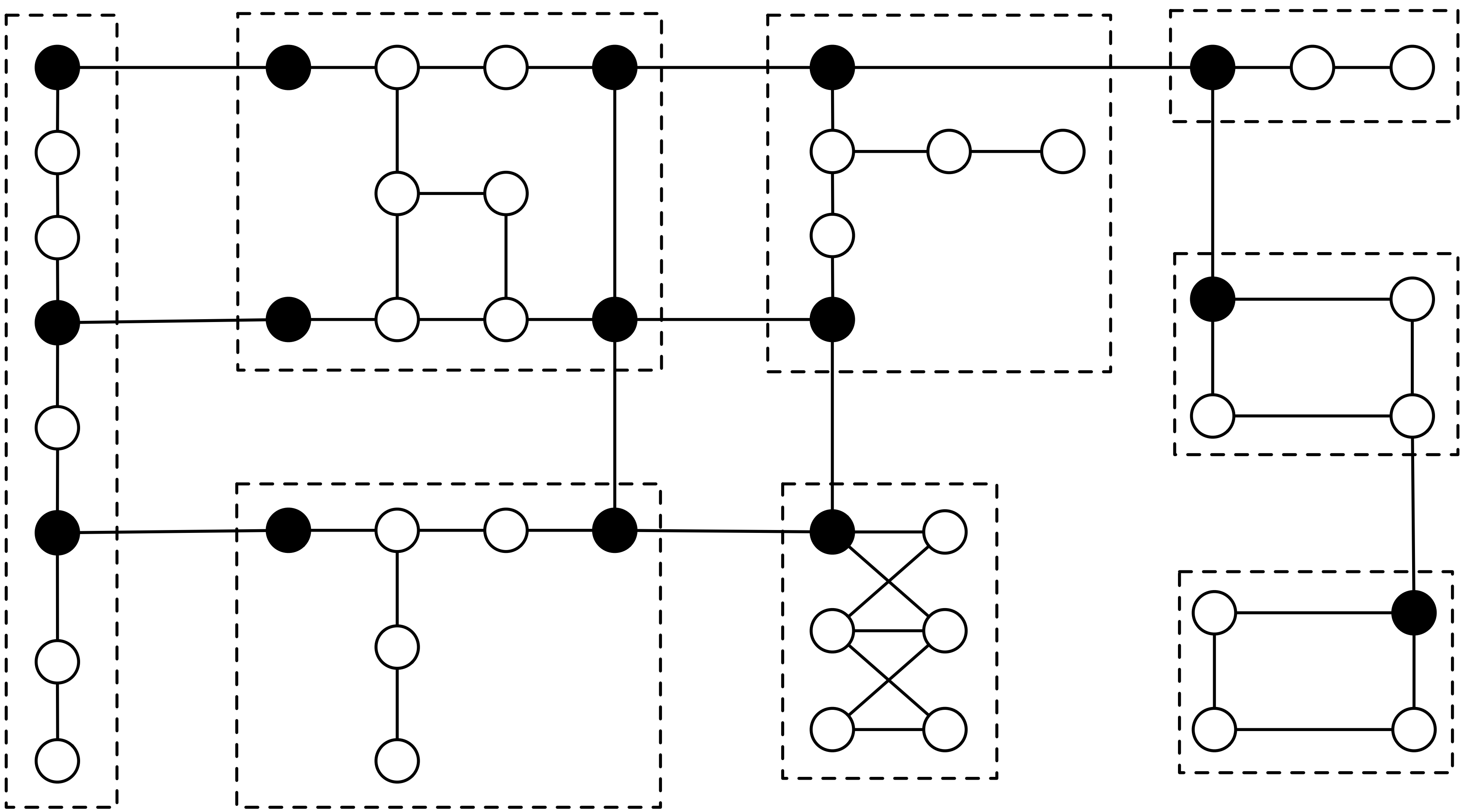
$$x_i(t+1) = \overbrace{A_i x_i(t)}^{\text{local dynamics}} + \overbrace{\sum_{j \in \mathcal{N}_i} A_{ij} x_j(t)}^{\text{interconnections}} + \overbrace{B_i u_i(t)}^{\text{local controls}},$$

where $i \in \{1, \dots, N\}$ and $\mathcal{N}_i := \{j \ : \ A_{ij} \neq 0\}$.

Decoupled control strategy

1. Partition the network into disjoint connected parts.

2. Select boundary nodes as control nodes.



Decoupled control strategy

3. Apply the inputs

$$u_i(t) := v_i(t) - \sum_{j \in \mathcal{N}_i} B_i^T A_{ij} x_j(t)$$

This control law yields N decoupled subsystems

$$x_i(t+1) = A_i x_i(t) + B_i v_i(t)$$

4. Choose v_i which minimizes the energy to steer the subsystem to the desired substate.

Local controllability

$$\Lambda := \begin{bmatrix} \lambda_{\min}^{-1}(W_{1,T}) & 0_2 & \cdots & 0 \\ 0 & \lambda_{\min}^{-1}(W_{2,T}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{\min}^{-1}(W_{N,T}) \end{bmatrix}.$$

Coupling strength

$$\Delta := \begin{bmatrix} 1 & \|A_{12}\|_2 & \cdots & \|A_{1N}\|_2 \\ \|A_{21}\|_2 & 1 & \cdots & \|A_{2N}\|_2 \\ \vdots & \vdots & \ddots & \vdots \\ \|A_{N1}\|_2 & \|A_{N2}\|_2 & \cdots & 1 \end{bmatrix}.$$

Theorem If we choose a decoupled control law then we obtain

$$\lambda_{\min}(W_T) \geq \frac{(1-\bar{\lambda}_{\max})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2}$$

where

$$\bar{\lambda}_{\max} = \max\{\lambda_{\max}(A_i) : i \in \{1, \ldots, N\}\} < 1$$

Controllers positioning

Theorem If we choose a decoupled control law
then we obtain

$$\lambda_{\text{min}}(W_T) \geq \frac{(1-\bar{\lambda}_{\text{max}})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2} = \frac{\text{local convergence speed}}{\text{(local controllability)(coupling strength)}}$$

where

$$\bar{\lambda}_{\text{max}} = \max\{\lambda_{\text{max}}(A_i) : i \in \{1, \ldots, N\}\} < 1$$

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where

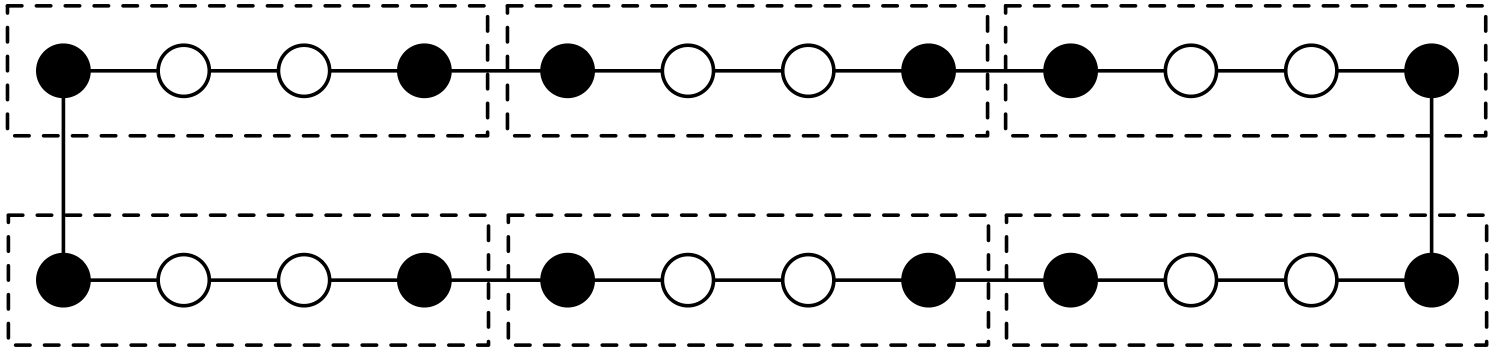
$$\bar{\lambda}_{\max} = \max\{\lambda_{\max}(A_i) : i \in \{1, \dots, N\}\} < 1$$

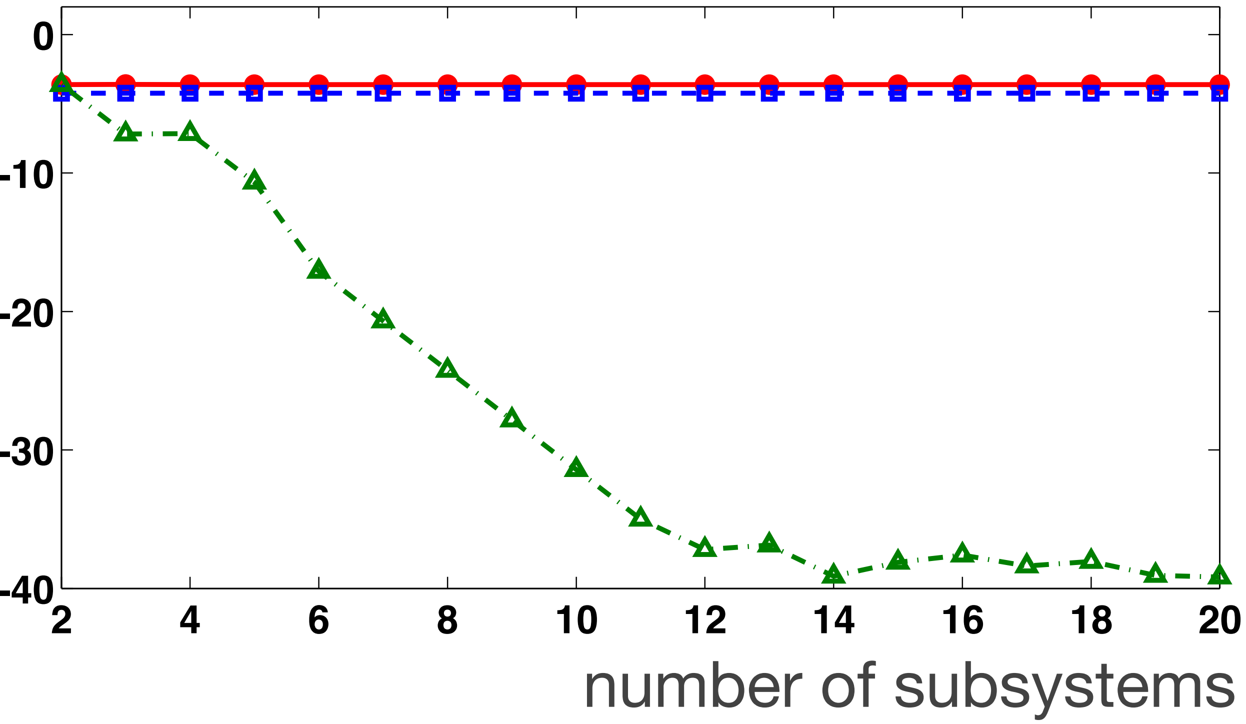
For high controllability degree:

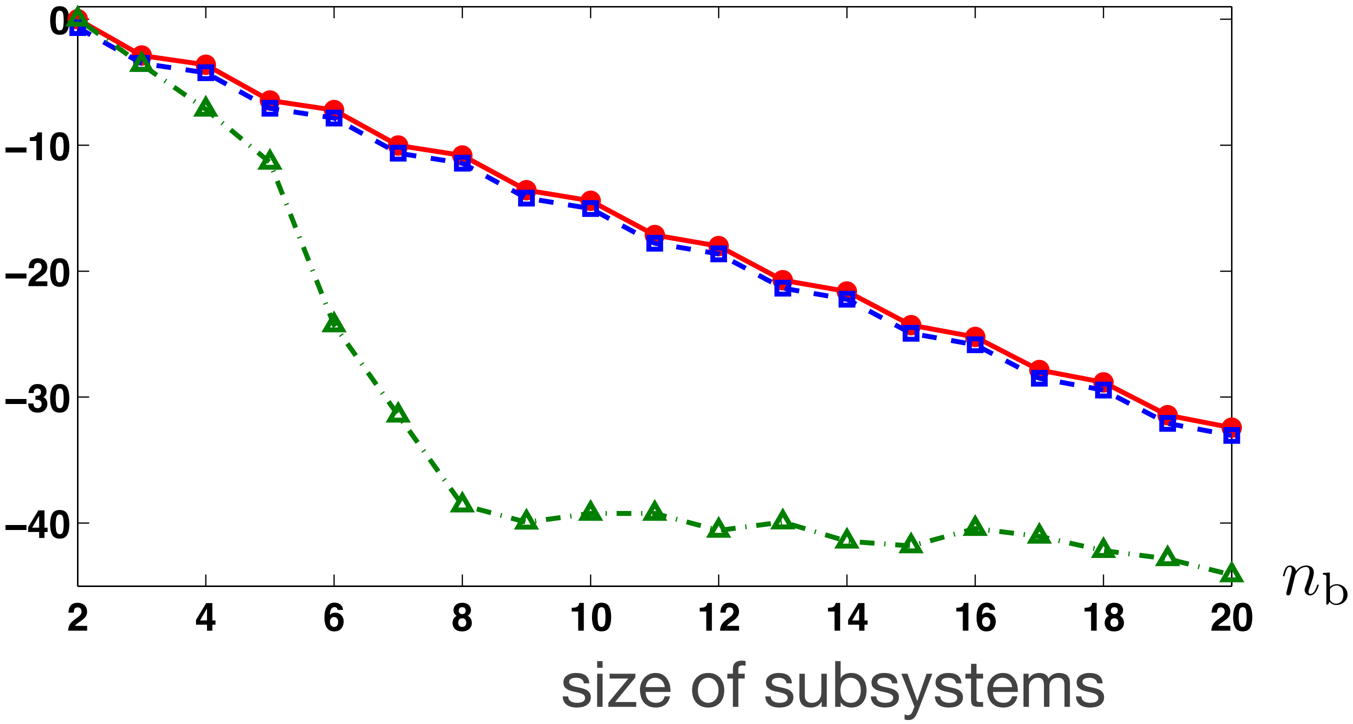
1. Partition so that $\|\Delta\|_\infty$ are small (weakly coupled subsystems)
2. Select local control nodes so that $\|\Lambda\|_\infty$ is small (large local controllability)

With decoupled control, global controllability becomes a local property

Circulant graph







λ_{min} with the decoupled control strategy

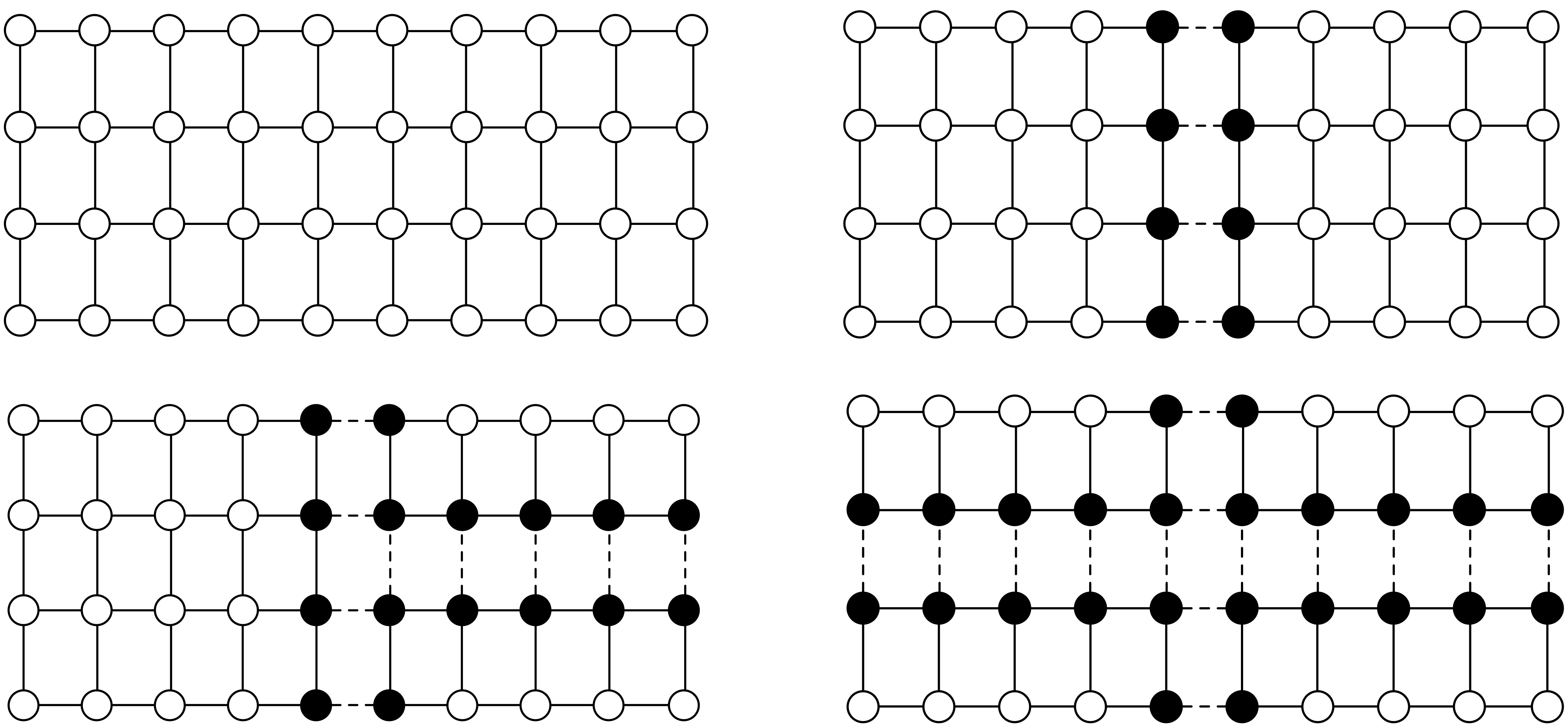
λ_{min} theoretical lower bound

λ_{min} with random positioning

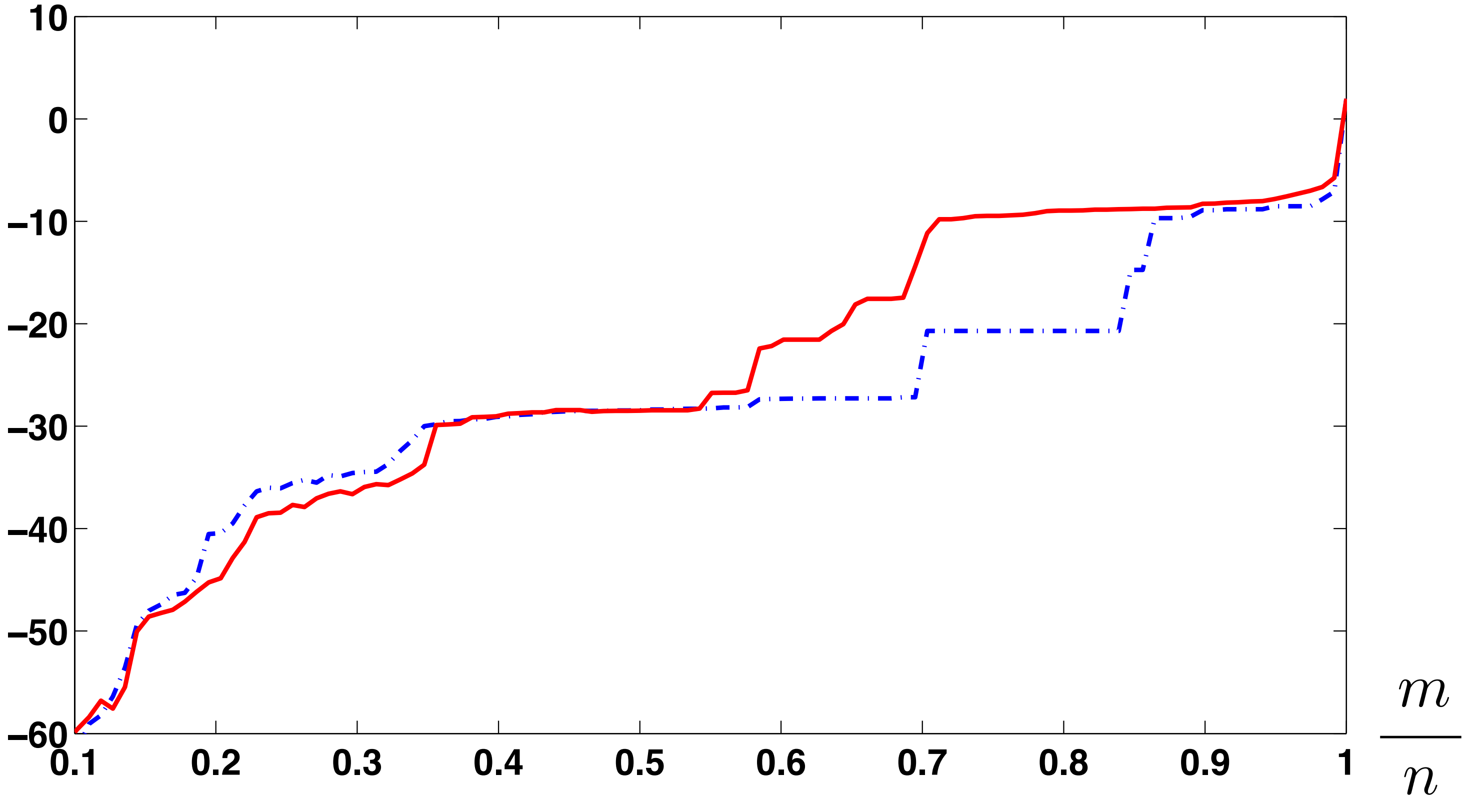
Selection of the control nodes

Until the desired number of control nodes have been selected:

1. Bisect the least controllable subsystem via Fiedler partitioning.
2. Include boundary nodes in the control set.



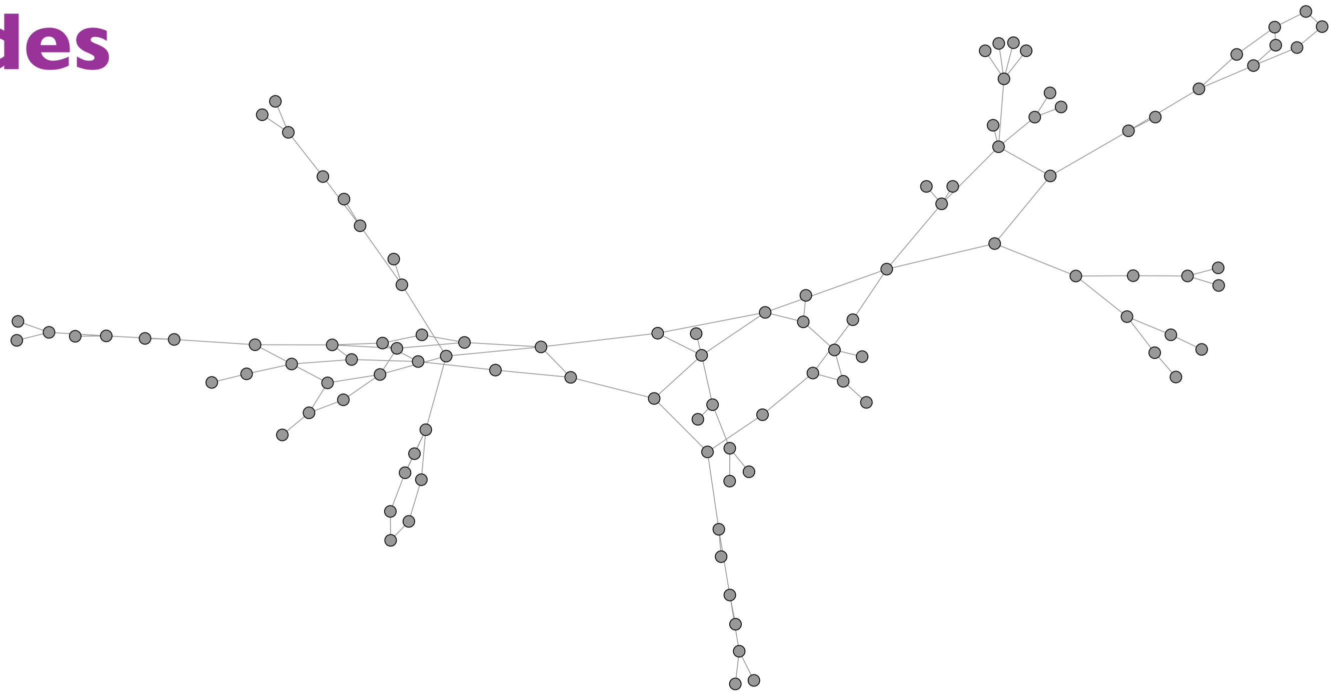
Power grid with 118 nodes

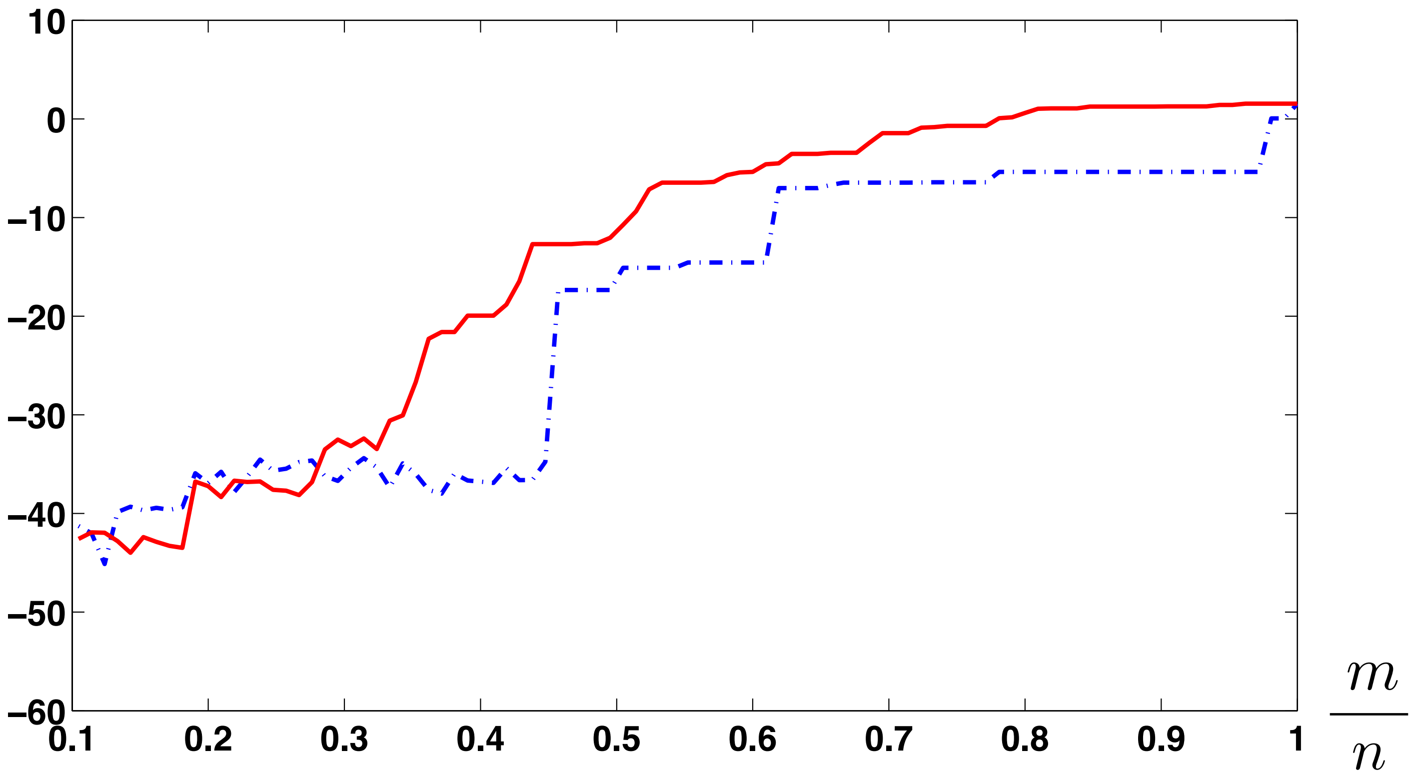


$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Epidemics network with 86 nodes

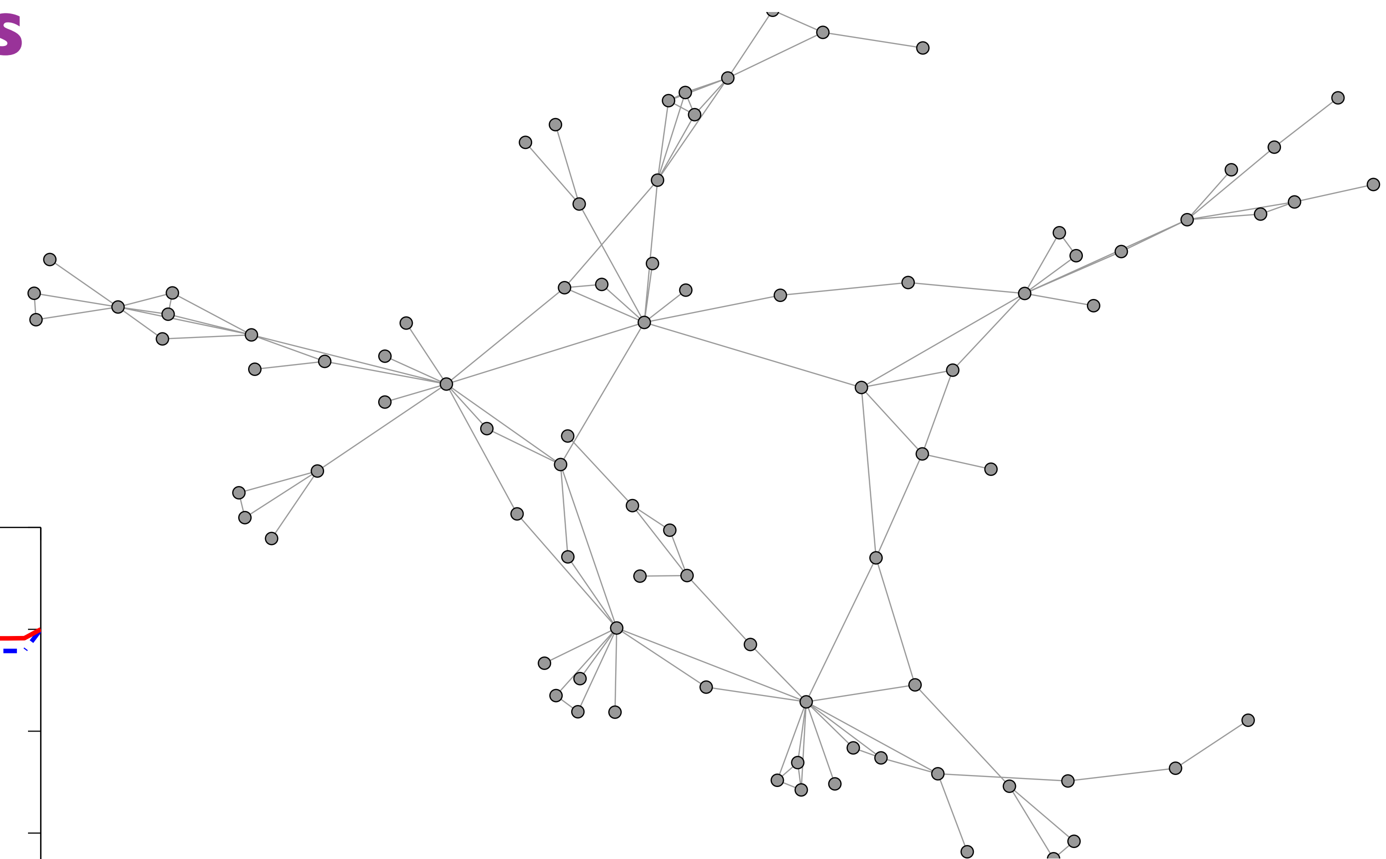


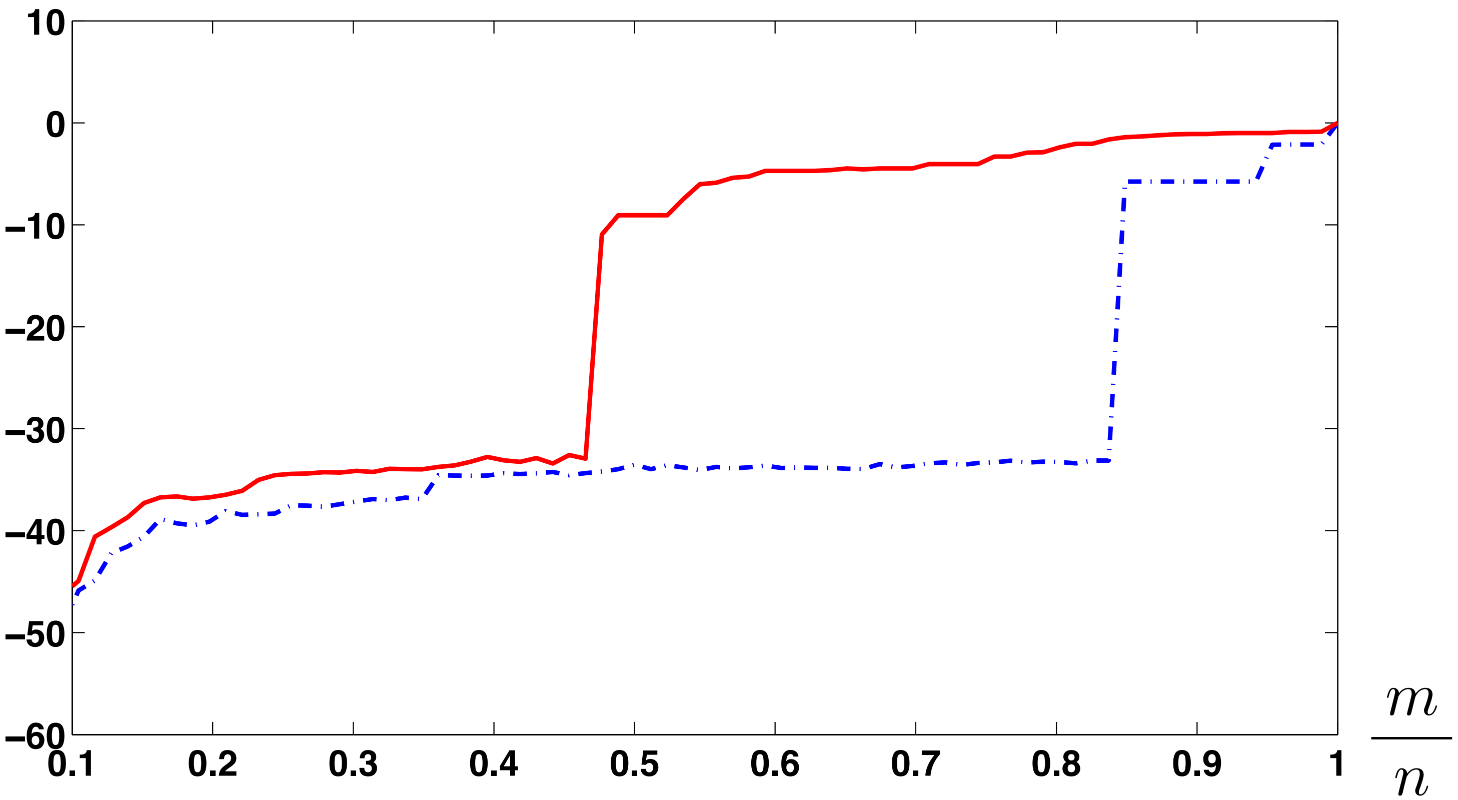


$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Social network with 105 nodes





$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

Conclusions

- Similar results for **observability**
- For symmetric (isotropic) networks we need to control **a fixed fraction** of nodes
- For anisotropic networks it is enough to control **a fixed number** of nodes
- **Random** positioning works pretty well
- **Phase transition** can be noticed (critical fraction of controlled nodes)
- There are a lot of open problems:
  - Understanding isotropic and anisotropic networks
  - Controllability of random and of structured graphs
  - Performance of random positioning
  - Phase transition
  - Different metrics

Conclusions

Controllability ↓ **Graph theory**

Controllability degree ↓ **Spectral graph theory**

Thank you