Differential positivity

Rodolphe Sepulchre -- University of Cambridge, UK

Lund, LCCC workshop, October 2014

The main reference

Differentially positive systems

F. Forni, R. Sepulchre

Abstract

The paper introduces and studies differentially positive systems, that is, systems whose linearization along an arbitrary trajectory is positive. A generalization of Perron Frobenius theory is developed in this differential framework to show that the property induces a (conal) order that strongly constrains the asymptotic behavior of solutions. The results illustrate that behaviors constrained by local order properties extend much beyond the well-studied class of linear positive systems and monotone systems, which both require a constant cone field and a linear state space.

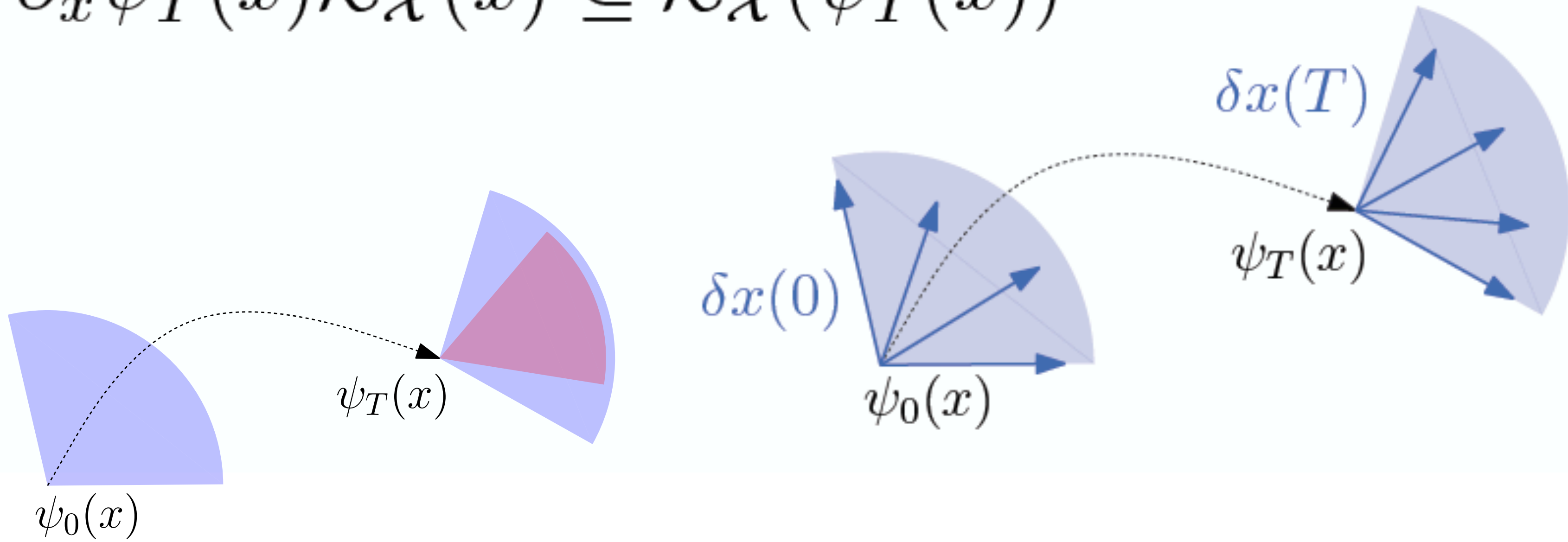
This talk: why did we study that property ?

Differential positivity

A linear map is positive if it leaves a cone invariant.

A dynamical system is differentially positive if its linearization along an arbitrary trajectory is positive.

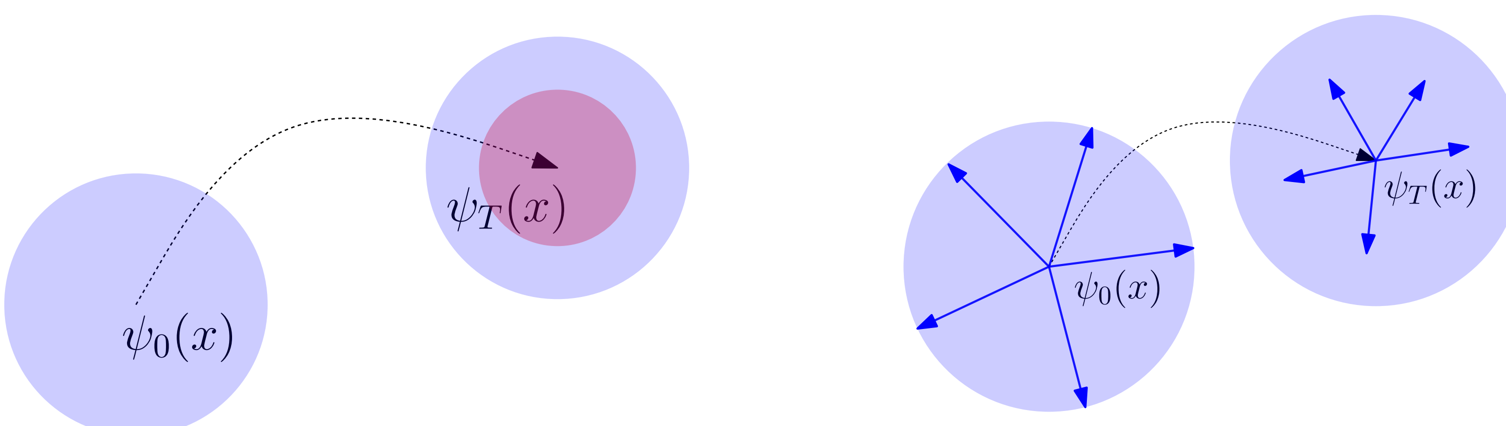
$$\partial_x \psi_T(x) \mathcal{K}_{\mathcal{X}}(x) \subseteq \mathcal{K}_{\mathcal{X}}(\psi_T(x))$$



Compared to contraction,
a cone replaces the ball...

A linear map is (Lyapunov) stable if it leaves a ball invariant.

A dynamical system is differentially stable (non expanding) if its linearization along an arbitrary trajectory is Lyapunov stable.



In systems and control: Lohmiller & Slotine (1998), and many others since then...
Terminology: contraction, convergence, incremental stability, ...

Textbook Perron-Frobenius theory

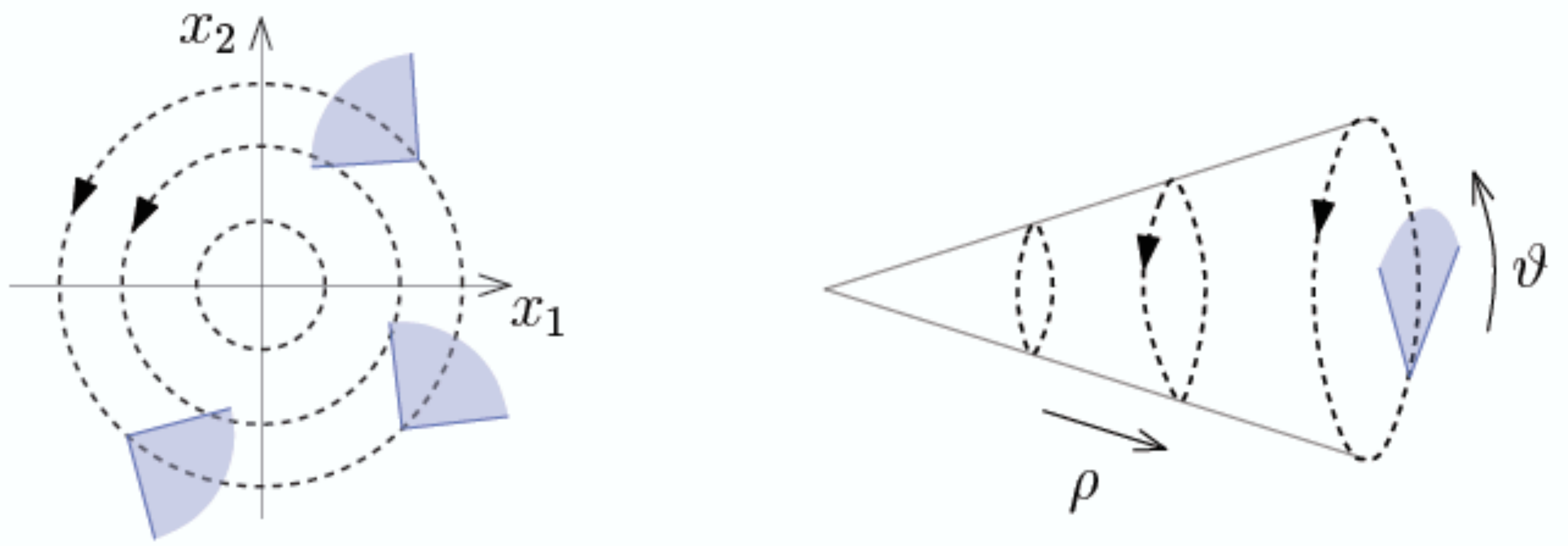
Perron–Frobenius theorem

From Wikipedia, the free encyclopedia

In linear algebra, the **Perron–Frobenius theorem**, proved by Oskar Perron (1907) and Georg Frobenius (1912), asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components, and also asserts a similar statement for certain classes of nonnegative matrices. This theorem has important applications to probability theory (ergodicity of Markov chains); to the theory of dynamical systems (subshifts of finite type); to economics (Okishio's theorem, Leontief's input-output model);[1] to demography (Leslie population age distribution model),[2] to Internet search engines[3] and even ranking of football teams.[4]

An obstacle to think of positivity as a *geometric* property?

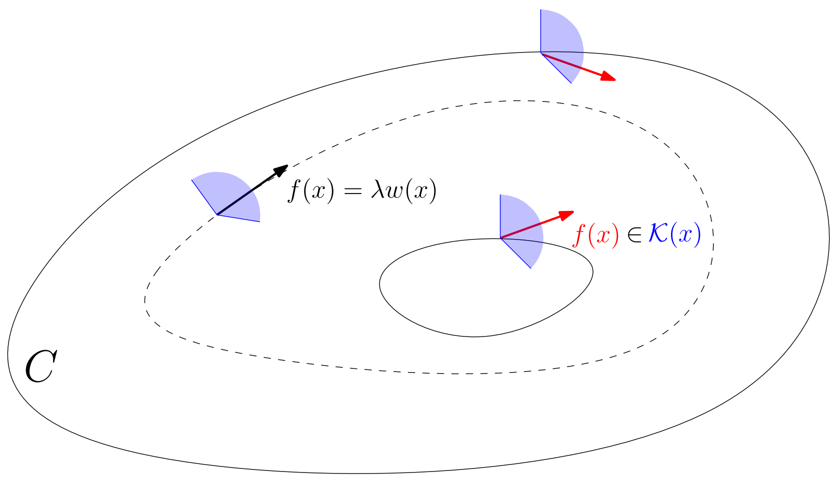
As a geometric concept, positivity is *not* antagonist to oscillations



$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases} \equiv \begin{cases} \dot{\rho} = 0 \\ \dot{\vartheta} = 1 \end{cases}$$

Strict differential positivity and nonlinear oscillations

Corollary 2: Under the assumptions of Theorem 3, consider an open, forward invariant region $\mathcal{C} \subseteq \mathcal{X}$ that does not contain any fixed point. If the vector field $f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$ for any $x \in \mathcal{C}$, then there exists a unique attractive periodic orbit contained in $\mathcal{C}$. ⌟



- Theorem 3 is a *differential* version of Perron-Frobenius theory.
- The corollary is akin to Poincare Bendixon theorem for planar systems.
- Strict differential positivity, similarly to the topology of the plane, enforces a one-dimensional asymptotic behavior.

Plan for this talk

- Motivation (1): positivity and networks
- Motivation (2): positivity and monotonicity
- Motivation (3): positivity on nonlinear spaces
- Motivation (4): positivity and interconnections

Recycling a few old slides...

Consensus theory and Hilbert metric

R. Sepulchre
University of Liege, Belgium

LCCC workshop
January 2010

Classical linear consensus theory

Linear consensus algorithms are linear time-varying systems

$$x(t+1) = A(t)x(t), \qquad x(t) \in \mathbb{R}^n$$

where for each t, A(t) is row stochastic, i.e.

A is nonnegative: $a_{ij} \geq 0$

each row sums to one: $A(t)1 = 1$

Uniform convergence to $\alpha 1$ ("consensus: $x_i = x_j$
is proven under uniform connectivity / irreducibility
(Tsitsiklis, Jadbabaie et al., Moreau, ...)

Convergence analysis and Lyapunov functions

Tsitsiklis (1986) observed that

$$V(x) = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i$$

is non increasing along the flow.

Uniform convergence is established by showing the strict decay of over a finite horizon.

It is known that no common quadratic Lyapunov exists in general.
(See *Olshevsky & Tsitsiklis 08* for a discussion)

Birkhoff Theorem

Let K a closed solid cone in X a Banach space, with partial ordering $\preceq$.

A is *positive* if A maps $\mathring{K}$ to $\mathring{K}$

A is *monotone* if $x \preceq y \Rightarrow Ax \preceq Ay$

Theorem (G. Birkhoff, 1957):

Positive linear monotone mappings contract the Hilbert metric in $\mathring{K}$.

The contraction coefficient is $\tanh \frac{1}{4} \Delta(A)$

Note: Perron-Frobenius follows from contraction mapping theorem

Conclusions

Conic geometries are adapted to consensus theory ...
Quadratic Lyapunov functions are'nt ...

Tsitsiklis Lyapunov function is a measure of contraction of the Hilbert metric.

Birkhoff theorem (positive monotone operators contract the Hilbert metric) applies to more general cones, e.g. the SDP cone.

Opens the way to a consensus theory in noncommutative spaces, with a number of possible applications.

How to bridge the gap between contraction measures and the i/o approach to consensus ?

Contraction analysis of linear consensus

Consider the displacements dynamics from (49) given by $\dot{\delta x} = A(t)\delta x$, and the horizontal Finsler-Lyapunov function

$$V(x, \delta x) := \max_i \delta x_i - \min_i \delta x_i, \quad (50)$$

that coincides with the classical consensus function adopted in [30], [53] lifted to the tangent space. See [43] for its

A differential Lyapunov framework for contraction analysis, F. Forni and RS, TAC 2014.

Consensus theory connects to contraction analysis by interpreting

- Hilbert metric as a *Finsler-Lyapunov function* to study contraction
- the consensus (=Perron-Frobenius) direction as a symmetry to be factored out in the contraction analysis
- Projective contraction + row-stochasticity as horizontal contraction

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Monotone Control Systems

David Angeli and Eduardo D. Sontag, *Fellow, IEEE*

I. Introduction

ONE OF THE most important classes of dynamical systems in theoretical biology is that of *monotone systems*. Among the classical references in this area are the textbook by Smith [27] and the papers [14] and [15] by Hirsh and [26] by Smale. Monotone systems are those for which trajectories preserve a partial ordering on states. They include the subclass of *cooper-*

Definition II.1: A controlled dynamical system $\phi : \mathbb{R}_{\geq 0} \times X \times \mathcal{U}_\infty \to X$ is *monotone* if the following implication holds for all $t \geq 0$:

$$u_1 \succeq u_2,\ x_1 \succeq x_2 \quad \Rightarrow \quad \phi(t, x_1, u_1) \succeq \phi(t, x_2, u_2).$$

Reading the paper to the end...

Remark VIII.3: Looking at cooperativity as a notion of "incremental positivity" one can provide an alternative proof of the infinitesimal condition for cooperativity, based on the positivity of the variational equation. Indeed, assume that each system (35) is a positive linear time-varying system, along trajectories of (1). Pick arbitrary initial conditions $\xi_1 \succeq \xi_2 \in X$ and inputs $u_1 \geq u_2$. Let $\Phi(h) := \phi(t, \xi_2 + h(\xi_1 - \xi_2), u_2 + h(u_1 - u_2))$. We have (see, e.g., [28, Th. 1]) that $\phi(t, \xi_1, u_1) - \phi(t, \xi_2, u_2) = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(h)dh = \int_0^1 z_h(t, \xi_1 - \xi_2, u_1 - u_2)dh$, where z_h denotes the solution of (35) when $(\partial f/\partial u)(x, u)$ and $(\partial f/\partial u)(x, u)$ are evaluated along $\phi(t, \xi_2 + h(\xi_1 - \xi_2), u_2 + h(u_1 - u_2))$. Therefore, by positivity, and monotonicity of the integral, we have $\phi(t, \xi_1, u_1) - \phi(t, \xi_2, u_2) \succeq 0$, as claimed. □

We remark that monotonicity with respect to other orthants corresponds to generalized positivity properties for linearizations, as should be clear by Corollary III.3.

APPENDIX A

The importance of

Monotone Dynamical Systems

M.W. Hirsch*   Hal Smith †

We will see that the long-term behavior of monotone systems is severely limited. Typical conclusions, valid under mild restrictions, include the following:

- If all forward trajectories are bounded, the forward trajectory of almost every initial state converges to an equilibrium
- There are no attracting periodic orbits other than equilibria, because every attractor contains a stable equilibrium.
- In $\mathbb{R}^3$, every compact limit set that contains no equilibrium is a periodic orbit that bounds an invariant disk containing an equilibrium.
- In $\mathbb{R}^2$, each component of any solution is eventually increasing or decreasing.

An analyst viewpoint on Perron-Frobenius theory

and nonlinear operators on Banach spaces. The usefulness of operators that are positive in some sense stems from the theorem of Perron [154] and Frobenius [48], now almost a century old, asserting that for a linear operator on $\mathbb{R}^n$ represented by a matrix with positive entries, the spectral radius is a simple eigenvalue having a positive eigenvector, and all other eigenvalues have smaller absolute value and only nonpositive eigenvectors. In 1912 Jentsch [84] proved the existence of a positive eigenfunction with a positive eigenvalue for a homogeneous Fredholm integral equation with a continuous positive kernel.

In 1935 the topologists Alexandroff and Hopf [2] reproved the Perron-Frobenius theorem by applying Brouwer's fixed-point theorem to the action of a positive $n \times n$ matrix on the space of lines through the origin in $\mathbb{R}^n_+$. This was perhaps the first explicit use of the dynamics of operators on a cone to solve an eigenvalue problem. In 1940 Rutman [169] continued in this vein by reproving Jentsch's theorem by means of Schauder's fixed-point theorem, also obtaining an infinite-dimensional analog of Perron-Frobenius, known today as the Krein-Rutman theorem [103, 213]. In 1957 G. Birkhoff [20] initiated the dynamical use of Hilbert's projective metric for such questions.

The dynamics of cone-preserving operators continues to play an important role in functional analysis; for a survey, see Nussbaum [145, 146]. One outgrowth of this work

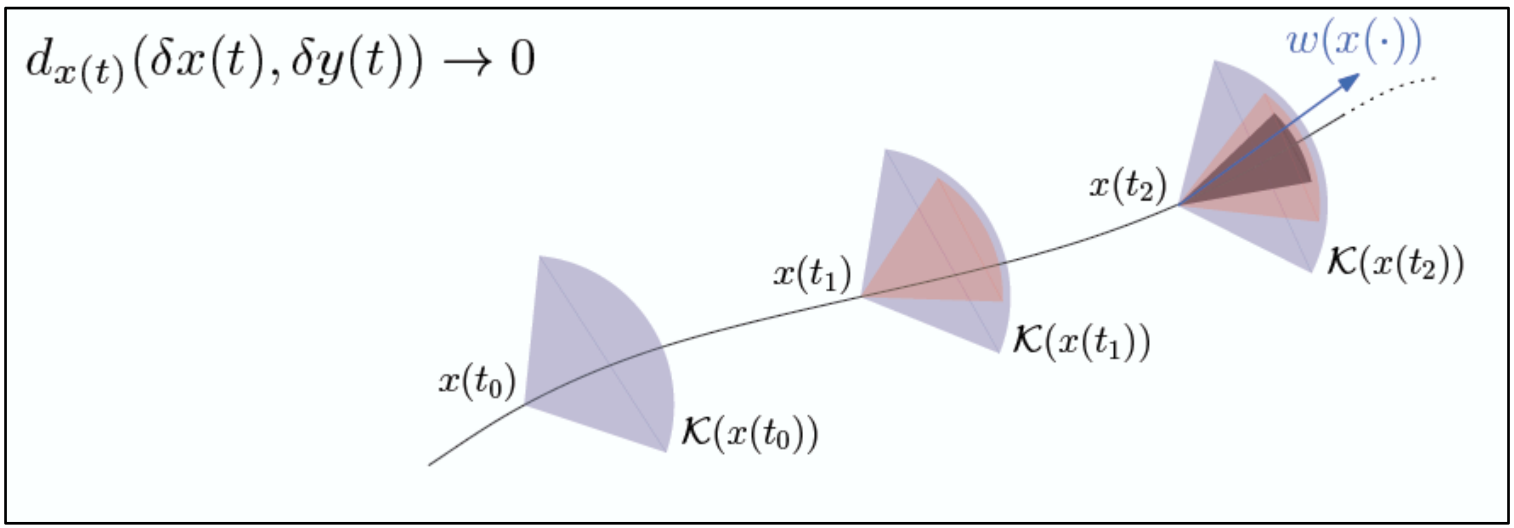
(*from* Hirsch and Smith, 2004)

A differential geometric viewpoint on PF theory

VI. Differential Perron-Frobenius theory

A. Contraction of the Hilbert metric

Bushell [10] (after Birkhoff [7]) used the Hilbert metric on cones to show that the strict positivity of a mapping guarantees contraction among the rays of the cone, opening the way to many contraction-based results in the literature of positive operators [10], [30], [39], [8], [26],



B. The Perron-Frobenius vector field

The Perron-Frobenius vector of a strictly positive linear map is a fixed point of the projective space. Its existence is a consequence of the contraction of the Hilbert metric, [10]. To exploit the

The main result: the PF vector field determines the asymptotic behavior

(Theorem 3)

Suppose that the trajectories of Σ are bounded. Then, for every $\xi \in \mathcal{X}$, the ω-limit set $\omega(\xi)$ satisfies one of the following two properties:

(i) The vector field $f(x)$ is aligned with the Perron-Frobenius vector field $w(x)$ for each $x \in \omega(\xi)$, and $\omega(\xi)$ is either a fixed point or a limit cycle or a set of fixed points and connecting arcs;

(ii) The vector field $f(x)$ is nowhere aligned with the Perron-Frobenius vector field $w(x)$ for each $x \in \omega(\xi)$, and either $\liminf_{t \to \infty} |\partial_x \psi(t, 0, x) w(x)|_{\psi(t,0,x)} = \infty$ or $\lim_{t \to \infty} f(\psi_t(x)) = 0$.

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Consensus on nonlinear spaces ☆

R. Sepulchre

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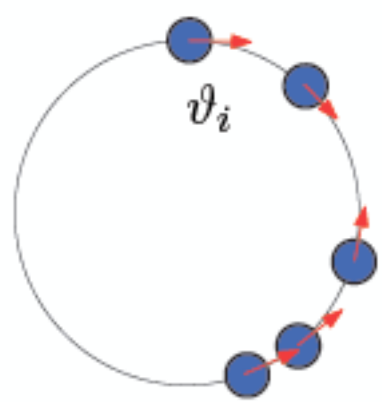
A B S T R A C T

Consensus problems have attracted significant attention in the control community over the last decade. They act as a rich source of new mathematical problems pertaining to the growing field of cooperative and distributed control. This paper is an introduction to consensus problems whose underlying state-space is not a linear space, but instead a highly symmetric nonlinear space such as the circle and other relevant generalizations. A geometric approach is shown to highlight the connection between several fundamental models of consensus, synchronization, and coordination, to raise significant global convergence issues not present in linear models, and to be relevant for a number of engineering applications, including the design of planar or spatial coordinated motions.



Differential positivity and consensus on nonlinear spaces

Current work:

Positivity is the local contraction property of the consensus rule

"move towards the average of your neighbors"

Inferring the cone field from the space geometry.

(Kuramoto model, phase synchronization, ...)

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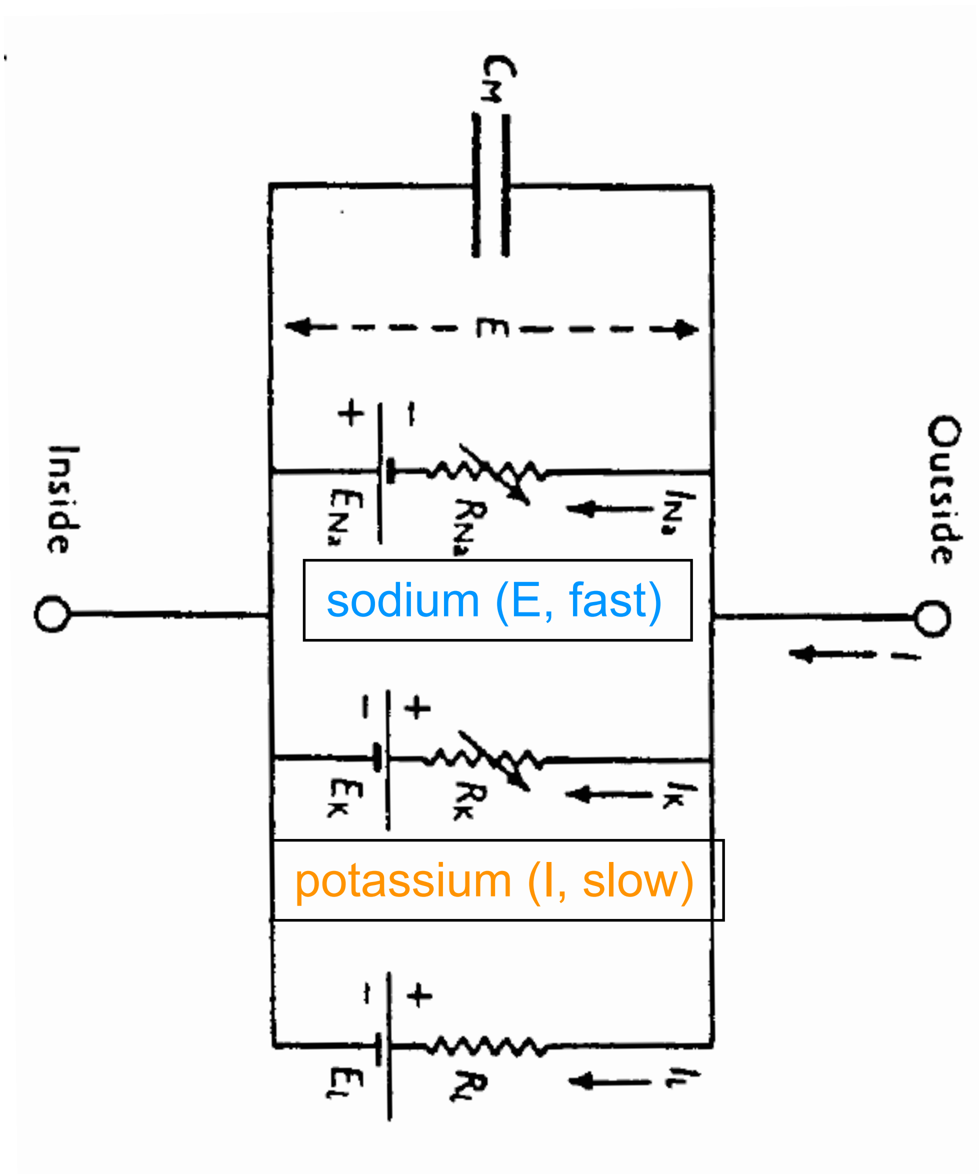
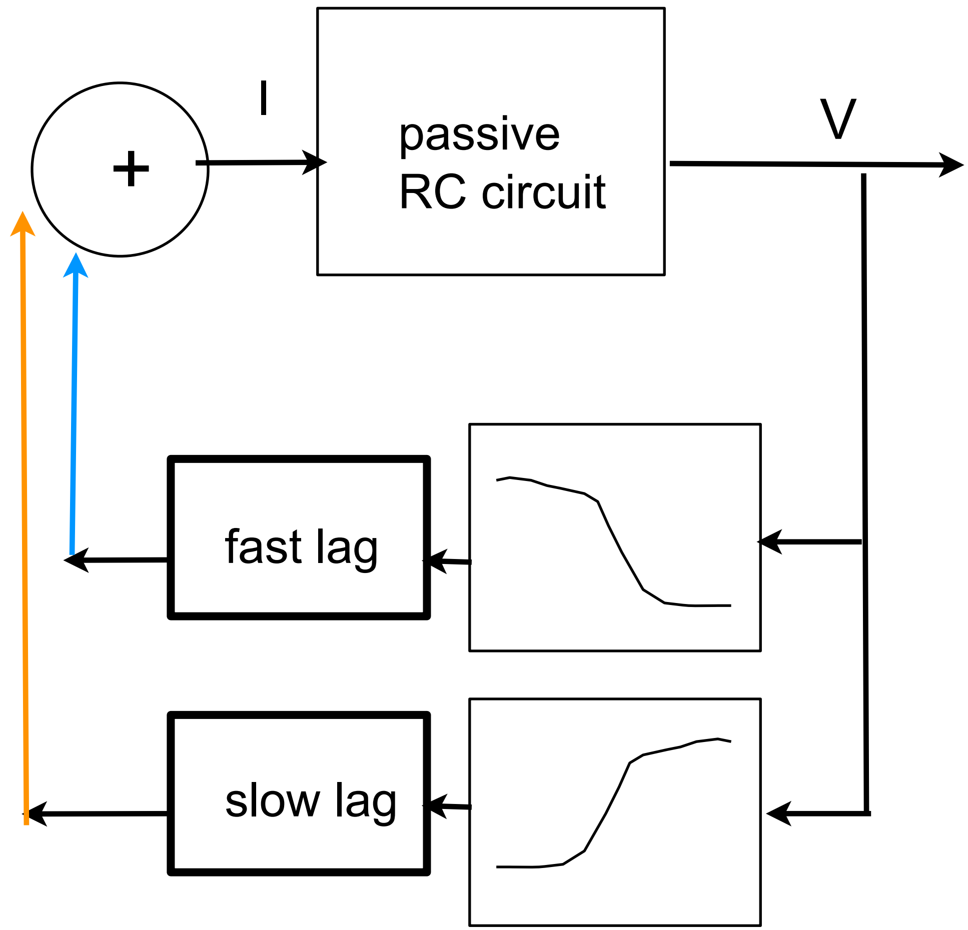
Hodgkin-Huxley electrical circuit is a mixed feedback interconnection of monotone systems





Differential positivity and interconnection of monotone systems

Current work:

Inferring the cone field from monotonicity of the blocks + the interconnection structure

(Negative feedback is not cone-preserving)

(Biological oscillators, bursters, ...)

Conclusion: differential positivity
=
smooth patching of local orders

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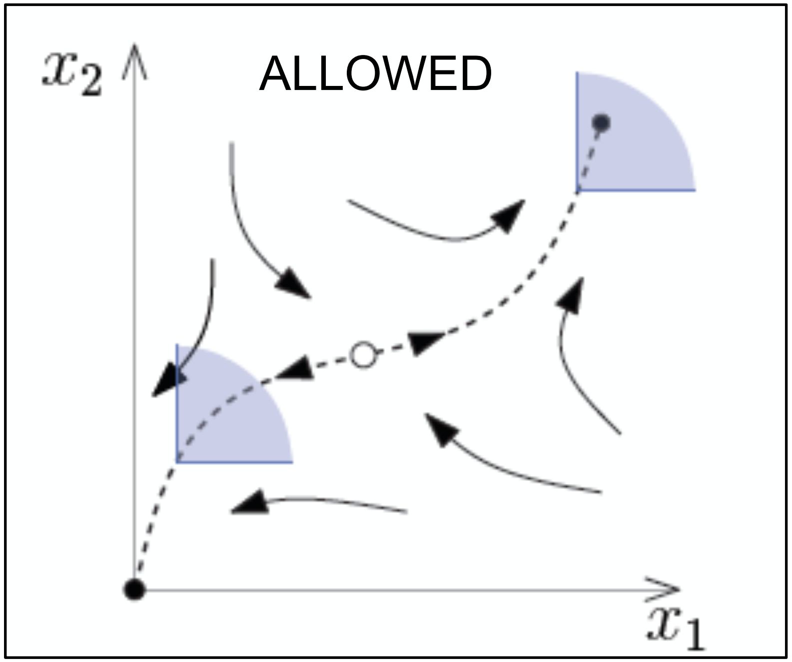
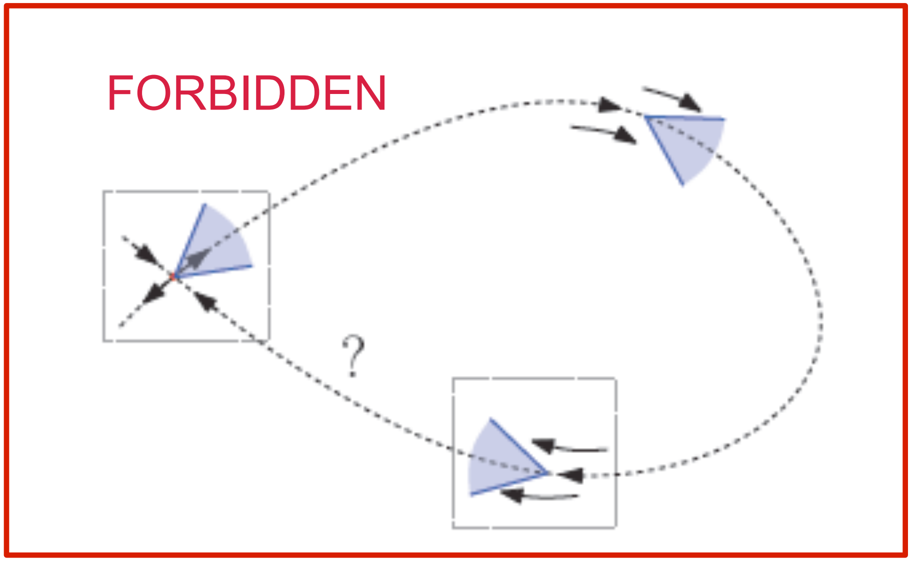
How weak is differential positivity ?





Local ordering is a weak property.
Smooth global patching is a demanding property.