Convex Relaxation of OPF

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Acknowledgment

Caltech

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Optimal power flow (OPF)

OPF is solved routinely to determine

- How much power to generate where
- Parameter setting, e.g. taps, VARs
- Market operation & pricing

Non-convex and hard to solve

- Huge literature since 1962
- Common practice: DC power flow (LP)
- Also: Newton-Ralphson, interior point, …

Outline

Optimal power flow (OPF)

- bus injection model, branch flow model

3 convex relaxations

- SDP, chordal, second-order cone (SOCP)
- Relation among them

Sufficient conditions for exact relaxation

- Radial: 3 main conditions
- Mesh: phase shifters

Summary: OPF (bus injection model)

$$\min \quad \operatorname{tr} CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \operatorname{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

nonconvex QCQP

Summary: OPF (branch flow model)

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s)$$

$$\text{s. t.} \quad \underline{s}_j \le s_j \le \overline{s}_j \qquad \underline{v}_j \le v_j \le \overline{v}_j$$

branch flow model

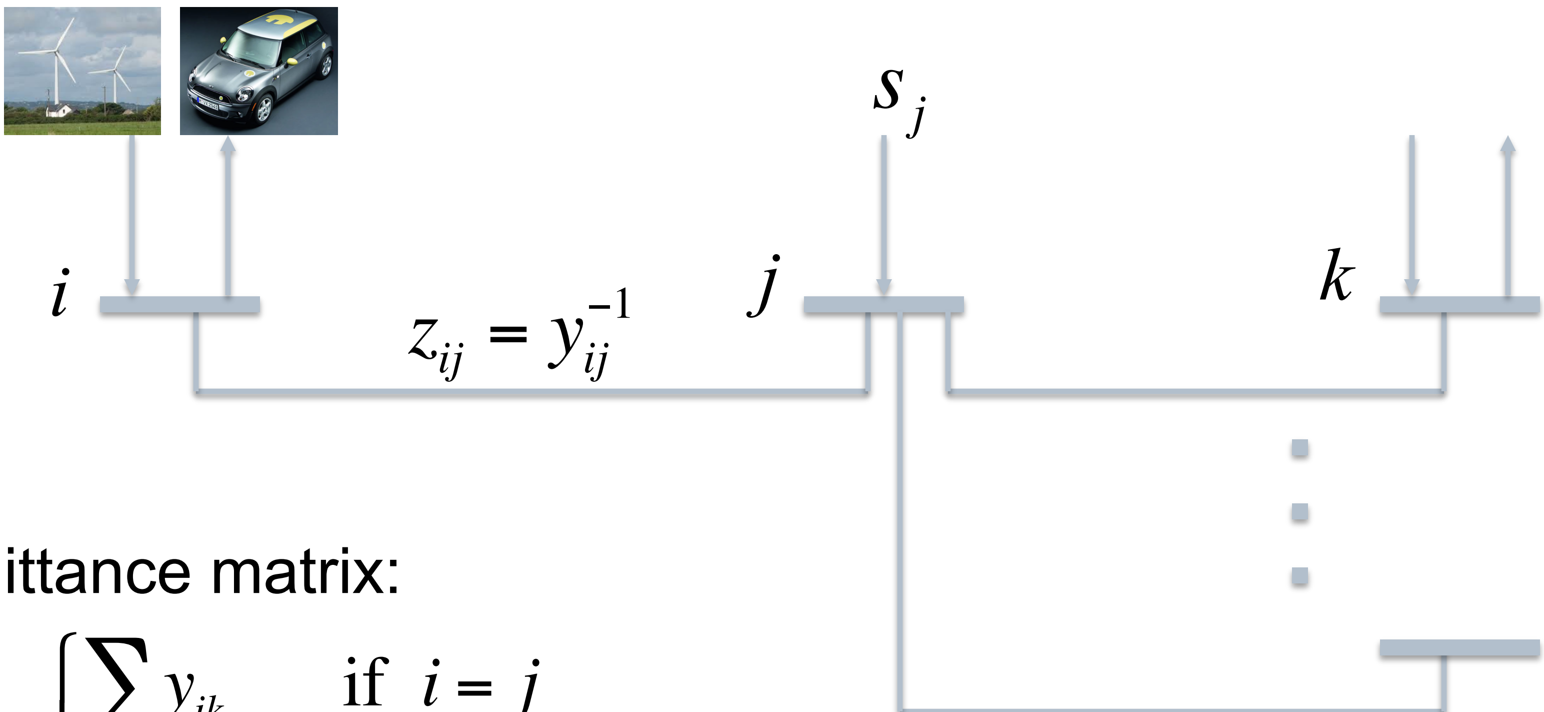
$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) - \sum_{j \to k} S_{jk} = s_j$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

nonconvex

details

Bus injection model



admittance matrix:

$$Y_{ij} := \begin{cases} \sum_{k \sim i} y_{ik} & \text{if } i = j \\ -y_{ij} & \text{if } i \sim j \\ 0 & \text{else} \end{cases}$$

graph model G: undirected

Y specifies topology of G and impedance z on lines

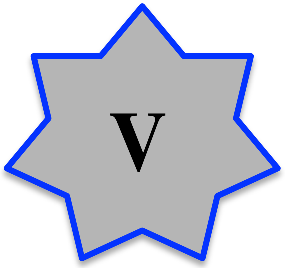
Bus injection model

In terms of V :

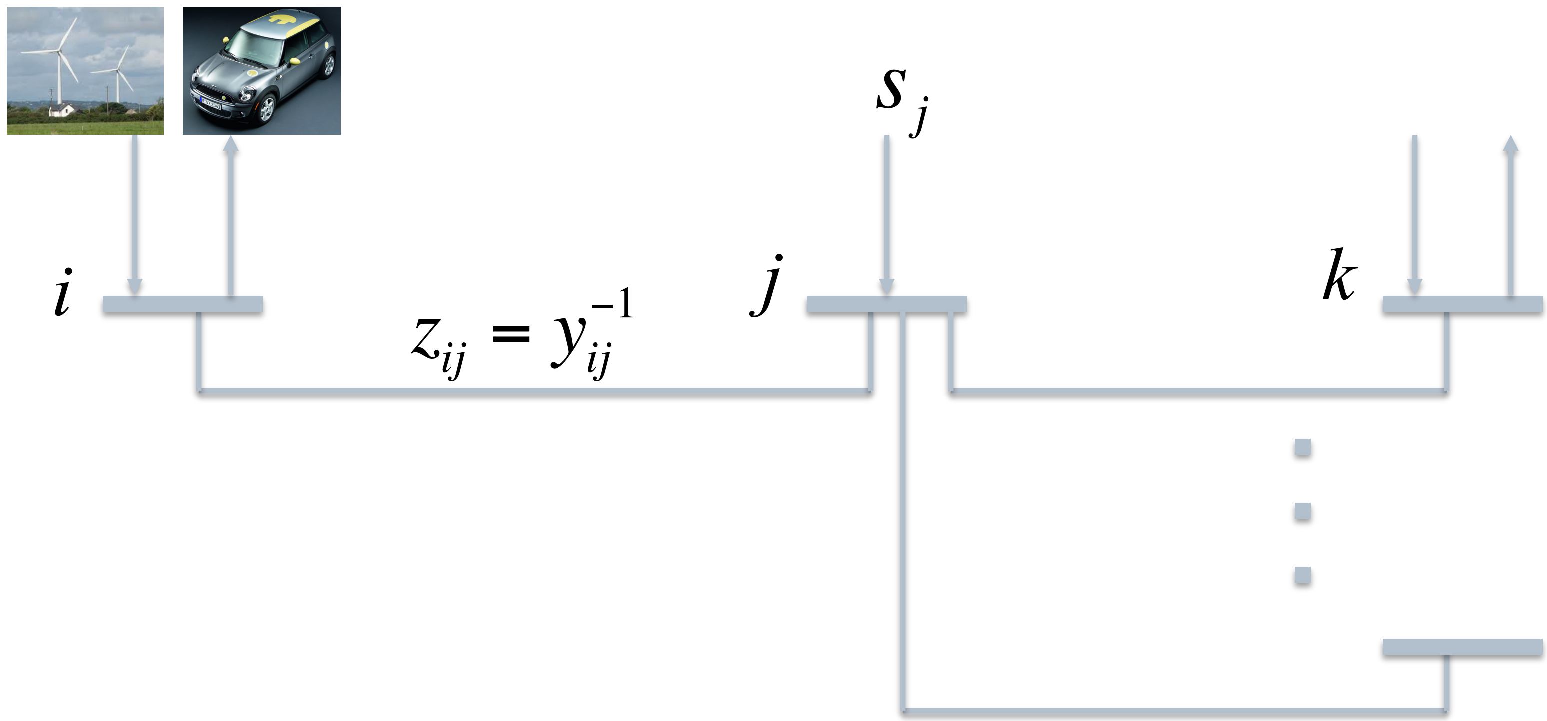
$$s_j = \operatorname{tr}\left(Y_j^H V V^H\right) \qquad \text{for all } j \qquad\qquad Y_j = Y^* e_j e_j^T$$

Power flow problem:

Given (Y, s) find V



Branch flow model



graph model G: directed

Branch flow model

$$V_i - V_j = z_{ij} I_{ij} \qquad \text{for all } i \rightarrow j$$ Kirchhoff law

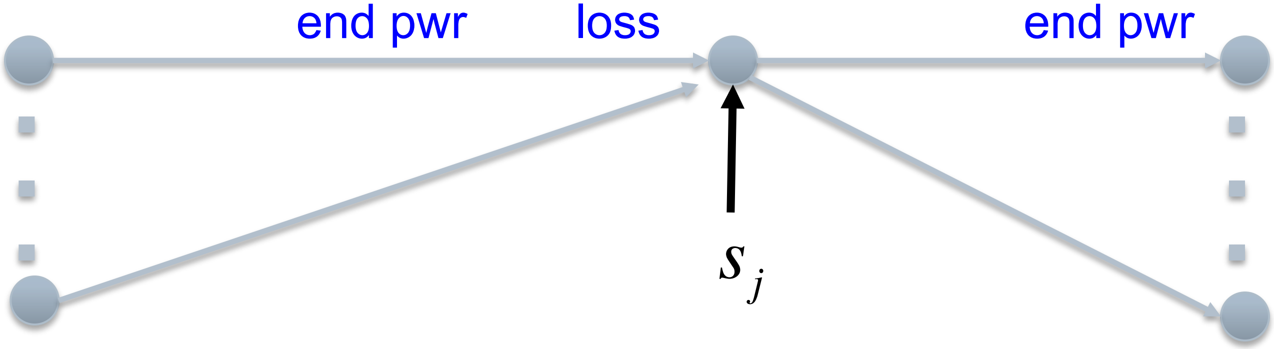
$$S_{ij} = V_i I_{ij}^* \qquad \text{for all } i \rightarrow j$$ power definition

$$\sum_{i \rightarrow j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j = \sum_{j \rightarrow k} S_{jk} \qquad \text{for all } j$$ power balance

sending end pwr

loss

sending end pwr



S_{ij} : branch power

I_{ij} : branch current

V_j : voltage

Branch flow model

$$V_i - V_j = z_{ij} I_{ij} \qquad \text{for all } \; i \rightarrow j$$ Kirchhoff law

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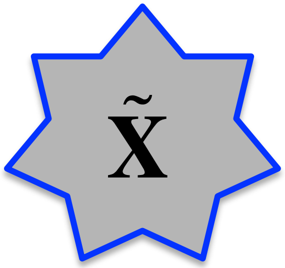
Power flow problem:

Given (z, s) find (S, I, V)



Recap

Bus injection model

$$s_j = \operatorname{tr}\left(Y_j V V^*\right)$$

$$(V, s) \in \mathbf{C}^{2(n+1)}$$

solution set

Branch flow model

$$V_i - V_j = z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^*$$

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$

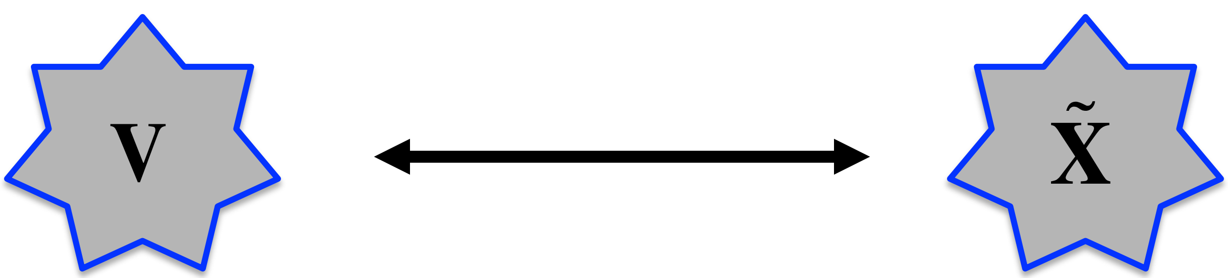
Equivalence

Theorem: $\mathbf{V} \equiv \tilde{\mathbf{X}}$

- BIM and BFM are equivalent in this sense
- Any result in one model is in principle provable in the other,
- … but some results are easier to formulate or prove in one than the other
- BFM seems to be much more numerically stable (radial networks)

$(V, s) \in \mathbf{C}^{2(n+1)}$ $\qquad$ $(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$

solution set



OPF: bus injection model

min	$V^* C V$	gen cost, power loss
over	(V, s)	
subject to	$\underline{s}_j \leq s_j \leq \overline{s}_j$	$\underline{V}_j \leq \lvert V_j \rvert \leq \overline{V}_j$

OPF: bus injection model

min $V^* C V$ — gen cost, power loss

over (V, s)

subject to $\underline{s}_j \leq s_j \leq \overline{s}_j \qquad \underline{V}_j \leq |V_j| \leq \overline{V}_j$

$$s_j = \mathrm{tr}\left(Y_j^H V V^H\right)$$

power flow equation

OPF: bus injection model

$$\min \quad \mathrm{tr}\, CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \mathrm{tr}\left(Y_j V V^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

quadratically constrained QP (QCQP)
nonconvex, NP-hard

OPF: branch flow model

$$\min \quad f(x)$$
$$\text{over} \quad x := (S, I, V, s)$$
$$\text{s. t.}$$

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$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

nonconvexity

Other features

Security constraint OPF

- Solve for operating points after each single contingency (N-1 security)
- N sets of variables and constraints, one for each contingency

Unit commitment

- Discrete variables

Stochastic OPF

- Chance constraints Pr(bad event) < ε

Other constraints

- Line flow, line loss, stability limit, …

… OPF in practice is a lot harder

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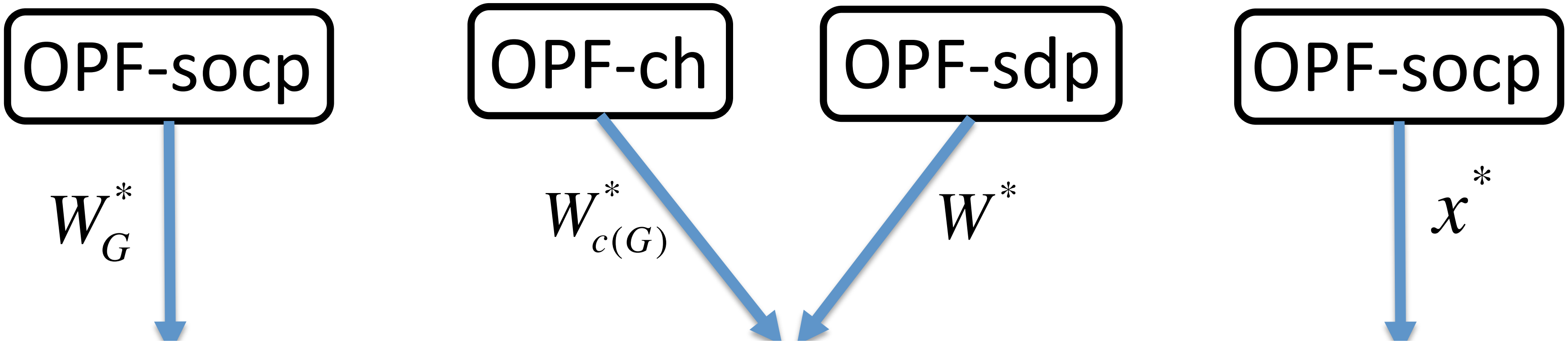

What are semidefinite relaxations of OPF?

How to check & recover global optimal ?

details

Literature

Convex relaxation of OPF

relaxation	model	first proposed	first analyzed
SOCP	BIM	Jabr 2006 TPS	
SDP	BIM	Bai et al 2008 EPES	Lavaei, Low 2012 TPS
Chordal	BIM	Bai, Wei 2011 EPES Jabr 2012 TPS	Molzahn et al 2013 TPS Bose et al 2014 TAC

Low. Convex relaxation of OPF (I, II), IEEE Trans Control of Network Systems, 2014

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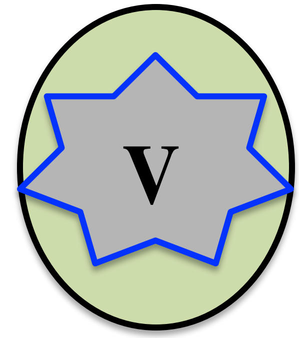
SOCP	BFM	Farivar et al 2011 SGC Farivar, Low 2013 TPS	Farivar et al 2011 SGC Farivar, Low 2013 TPS

Low. Convex relaxation of OPF (I, II), IEEE Trans Control of Network Systems, 2014

Basic idea



$$\min \quad \mathrm{tr}\, CVV^*$$

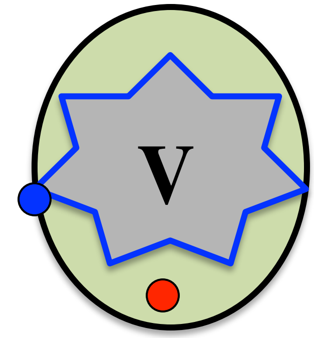
$$\text{subject to} \quad \underbrace{\underline{s}_j \le \mathrm{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j}_{\mathbf{V}}$$

All complexity due to nonconvexity of $\mathbf{V}$

Relaxations:

- design convex supersets of $\mathbf{V}$
- minimize cost over convex supersets

Basic idea



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All complexity due to nonconvexity of $\mathbf{V}$

Relaxations:

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- minimize cost over convex supersets

Exact relaxation: optimal solution of relaxation happens to lie in $\mathbf{V}$ (when?)

Basic idea

$$\min \quad \mathrm{tr}\, CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \mathrm{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{V}}$$

Approach

1. Three equivalent characterizations of $\mathbf{V}$
2. Each suggests a lift and relaxation

- What is the relation among different relaxations ?
- When will a relaxation be exact ?

Feasible sets

$$\min \quad \operatorname{tr} CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \operatorname{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

quadratic in V
linear in W

Equivalent problem:

$$\min \quad \operatorname{tr} CW$$

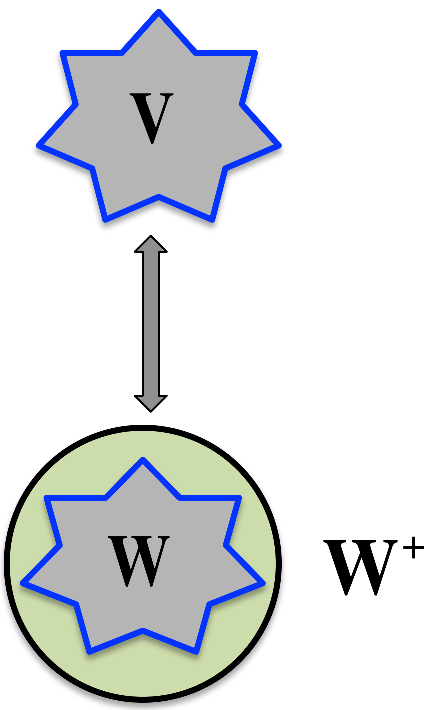
$$\text{subject to} \quad \underline{s}_j \le \operatorname{tr}\left(Y_j W\right) \le \bar{s}_j \qquad \underline{v}_i \le W_{ii} \le \bar{v}_i$$

$$W \ge 0, \quad \operatorname{rank} W = 1$$

convex in W
except this constraint

Equivalent feasible sets

$\mathbf{V} := \{V: \text{satisfies } \underline{\text{quadratic}} \text{ constraints}\}$



instead of n variables
solve for n^2 vars !

$\mathbf{W} := \{W: \text{satisfies } \underline{\text{linear}} \text{ constraints}\} \cap \{W \geq 0$ ~~rank-1~~ $\}$

idea: $W = VV^*$

Feasible set

only $n+2m$ vars !

linear in (W_{jj}, W_{jk}) ⟵ W_{jj} W_{jk}

$$\underbrace{\sum_{k:k\sim j} y_{jk}^* \left(|V_j|^2 - V_j V_k^* \right)}_{\text{linear in } (W_{jj},\, W_{jk})} : \ \textbf{only} \ \underbrace{|V_j|^2}_{W_{jj}} \text{ and } \underbrace{V_j V_k^*}_{W_{jk}}$$

corresponding to edges (j,k) in G!

$$\begin{array}{ll} \min & \operatorname{tr} CVV^* \\ \text{subject to} & \underbrace{\underline{s}_j \ \le \ \operatorname{tr}\left(Y_j VV^*\right) \ \le \ \bar{s}_j \qquad \underline{v}_j \ \le \ |V_j|^2 \ \le \ \bar{v}_j}_{\mathbf{V}} \end{array}$$

Feasible set

only $n+2m$ vars !

linear in (W_{jj}, W_{jk}) ⟵ W_{jj} W_{jk}

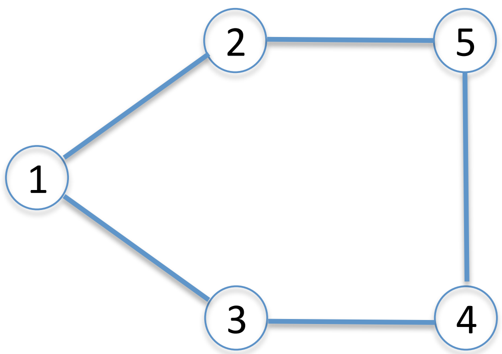
$$\sum_{k:k\sim j} y_{jk}^* \left(|V_j|^2 - V_j V_k^* \right) : \textbf{ only } |V_j|^2 \text{ and } V_j V_k^*$$

partial matrix W_G defined on G

$$W_G := \left\{ [W_G]_{jj}, [W_G]_{jk}, [W_G]_{kj} \,\middle|\, j, jk \in G \right\}$$

Kirchhoff's laws depend directly only on W_G

Example



G

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} \\ W_{41} & W_{42} & W_{43} & W_{44} & W_{45} \\ W_{51} & W_{52} & W_{53} & W_{54} & W_{55} \end{bmatrix}$$

SDP solves for $W \in \mathbf{C}^{n^2}$

n^2 variables

$$W_G = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & & & W_{25} \\ W_{31} & & W_{33} & W_{34} & \\ & & W_{43} & W_{44} & W_{45} \\ & W_{52} & & W_{54} & W_{55} \end{bmatrix}$$

Want to solve for W_G

$n+2m$ variables

Feasible sets

OPF $$\mathbf{V} := \left\{ V \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j V V^*\right) \le \bar{s}_j, \quad \underline{v}_j \le |V_j|^2 \le \bar{v}_j \right\}$$

SDP

$$\mathbf{W} := \left\{ W \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j W\right) \le \bar{s}_j, \ \underline{v}_j \le W_{jj} \le \bar{v}_j \right\} \cap \left\{ W \ge 0, \mathrm{rank}\text{-}1 \right\}$$

depend only on W_G

depend on all entries of W

Feasible sets

OPF $\mathbf{V} := \left\{ V \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j V V^*\right) \le \bar{s}_j, \quad \underline{v}_j \le |V_j|^2 \le \bar{v}_j \right\}$

SDP

$$\mathbf{W} := \left\{ W \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j W\right) \le \bar{s}_j, \ \underline{v}_j \le W_{jj} \le \bar{v}_j \right\} \cap \left\{ W \ge 0, \text{rank-1} \right\}$$

first idea:

$$\mathbf{W}_G := \left\{ W_G \,\middle|\, \underline{s}_j \le \mathrm{tr}\left(Y_j W_G\right) \le \bar{s}_j, \ \underline{v}_j \le [W_G]_{jj} \le \bar{v}_j \right\} \cap \left\{ W_G \ge 0, \text{rank-1} \right\}$$

W_G is equivalent to V when G is chordal
Not equivalent otherwise

Equivalent feasible sets

$$\mathbf{W}_G := \left\{ \begin{matrix} W_{jj}, W_{jk} : (j,k) \text{ in } G \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{matrix} \right\} \cap \left\{ \begin{matrix} W(j,k) \geq 0 \text{ rank-1}, \\ \text{cycle cond on } \angle W_{jk} \end{matrix} \right\}$$

idea: $W_G = \left(VV^* \text{ only on } G \right)$

$$\mathbf{W}_{c(G)} := \left\{ \begin{matrix} W_{jj}, W_{jk} : (j,k) \text{ in } c(G) \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{matrix} \right\} \cap \left\{ W_{c(G)} \geq 0 \text{ rank-1} \right\}$$

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matrix completion [Grone et al 1984]

$$\mathbf{W} := \left\{ W : \text{satisfies } \underline{\text{linear}} \text{ constraints} \right\} \cap \left\{ W \geq 0 \text{ rank-1} \right\}$$

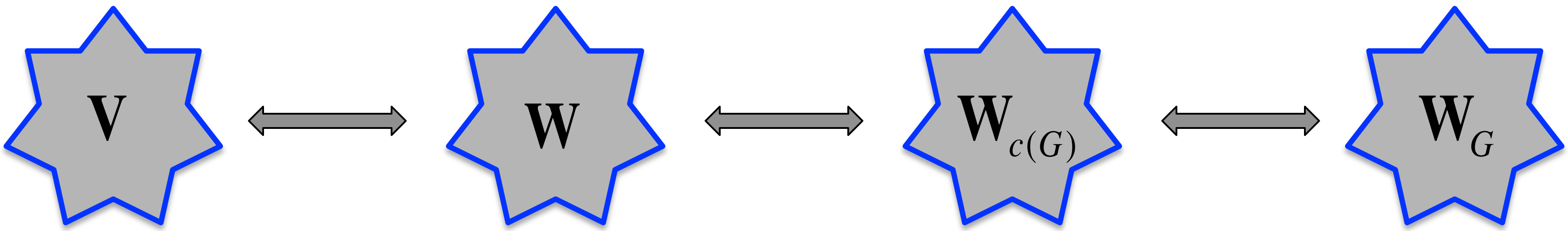
idea: $W = VV^*$

Cycle condition

local $$W_G(j,k) \succeq 0, \ \text{rank}\ W_G(j,k) = 1, \quad (j,k) \in E$$

global $$\sum_{(j,k)\in c} \angle [W_G]_{jk} = 0 \mod 2\pi$$ ← cycle cond

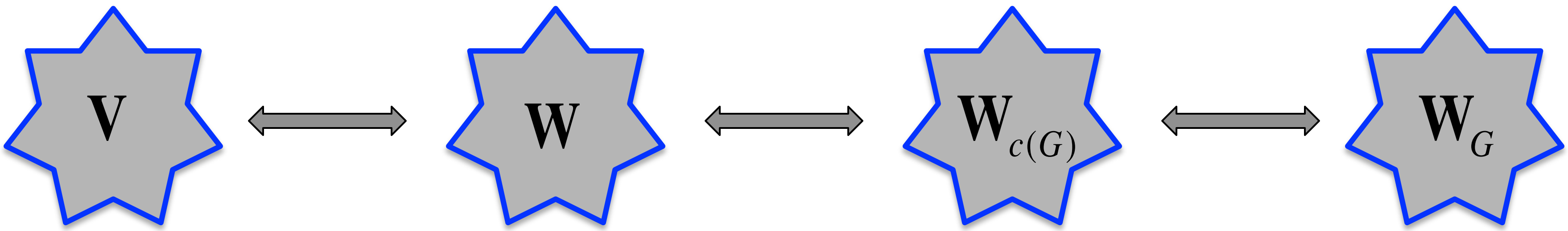
Equivalent feasible sets



Theorem: $\mathbf{V} \equiv \mathbf{W} \equiv \mathbf{W}_{c(G)} \equiv \mathbf{W}_G$

Bose, Low, Chandy Allerton 2012
Bose, Low, Teeraratkul, Hassibi TAC2014

Equivalent feasible sets



Theorem: $\mathbf{V} \equiv \mathbf{W} \equiv \mathbf{W}_{c(G)} \equiv \mathbf{W}_G$

Given $W_G \in \mathbf{W}_G$ or $W_{c(G)} \in \mathbf{W}_{c(G)}$ there is unique completion $W \in \mathbf{W}$ and unique $V \in \mathbf{V}$

Can minimize cost over **any** of these sets, but …

Relaxations

$$\mathbf{W}_G := \left\{ \begin{array}{l} W_{jj}, W_{jk} : (j,k) \text{ in } G \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{array} \right\} \cap \left\{ \begin{array}{l} W(j,k) \geq 0 \ \cancel{\text{rank-1}}, \\ \cancel{\text{cycle cond on } \angle W_{jk}} \end{array} \right\}$$

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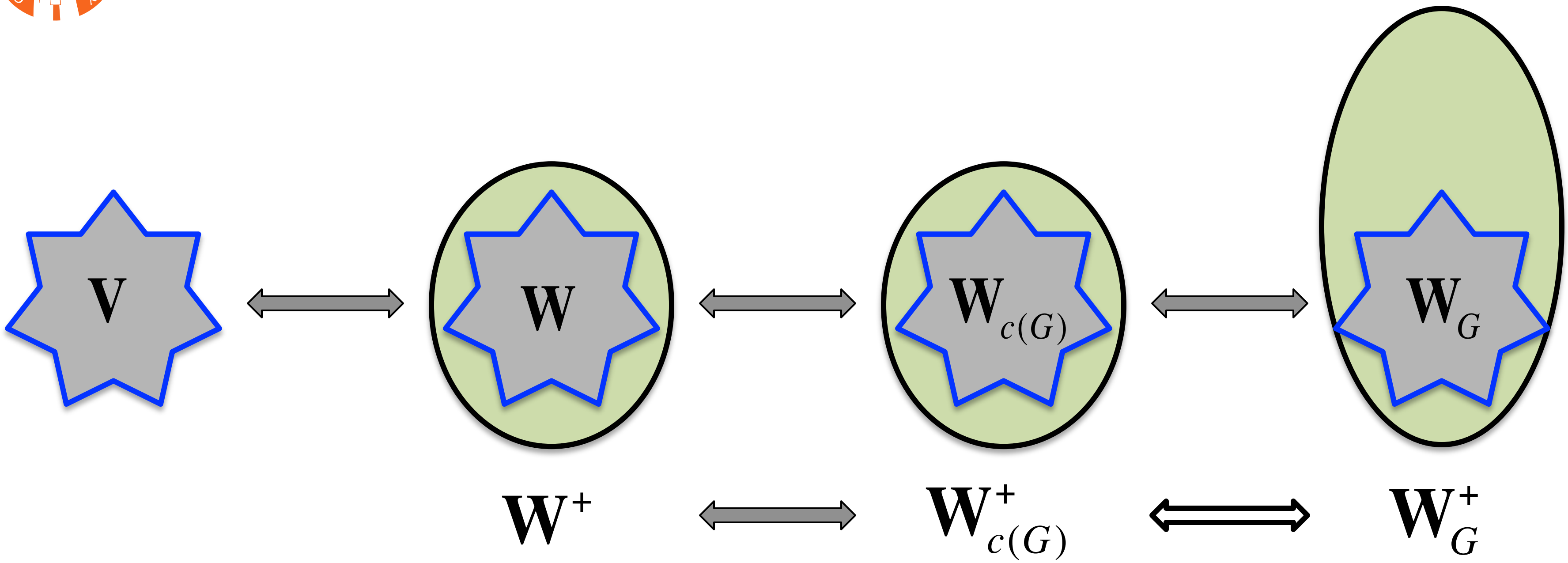
Relaxations



Theorem

- Radial G : $\mathbf{V} \subseteq \mathbf{W}^+ \cong \mathbf{W}^+_{c(G)} \cong \mathbf{W}^+_G$
- Mesh G : $\mathbf{V} \subseteq \mathbf{W}^+ \cong \mathbf{W}^+_{c(G)} \subseteq \mathbf{W}^+_G$

Bose, Low, Chandy Allerton 2012
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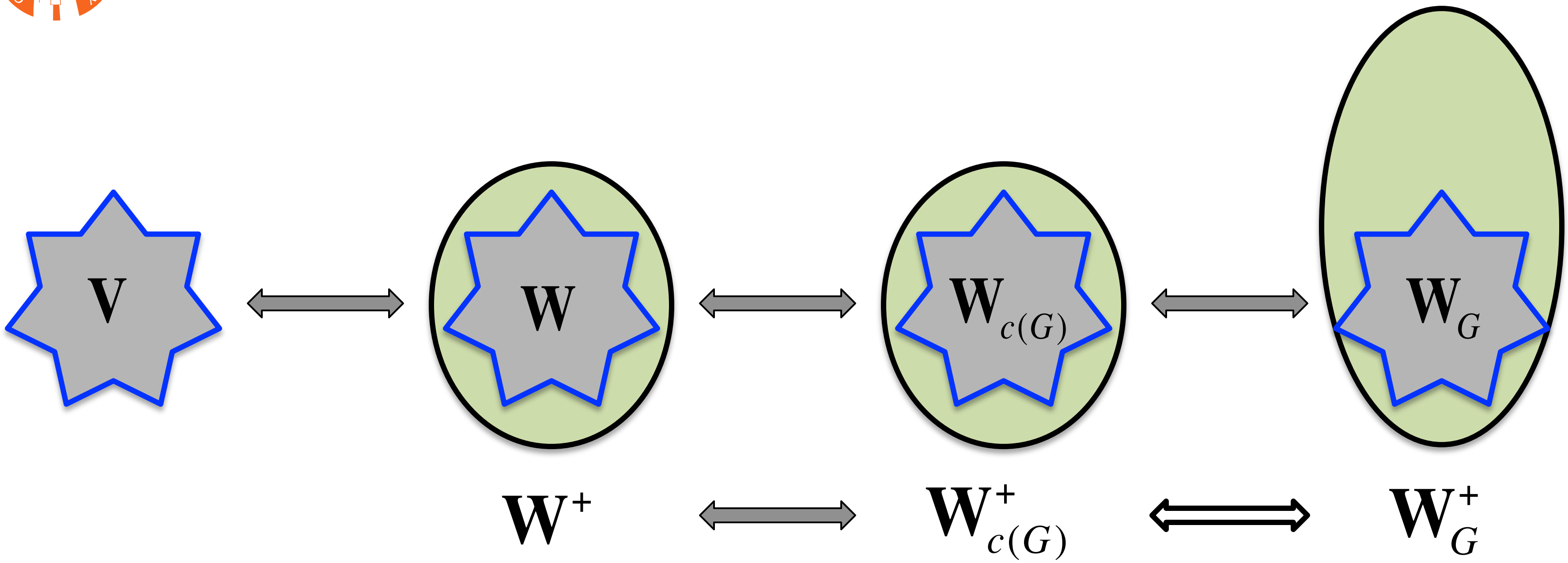
Relaxations



<u>Theorem</u>

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- Mesh G : $\mathbf{V} \subseteq \mathbf{W}^+ \cong \mathbf{W}^+_{c(G)} \subseteq \mathbf{W}^+_G$

For radial networks: always solve SOCP !

Convex relaxations

OPF

$$\min_{V} \; C(V) \quad \text{subject to} \;\; V \in \mathbf{V}$$

OPF-sdp:

$$\min_{W} C(W_G) \quad \text{subject to} \quad W \in \mathbb{W}^+$$

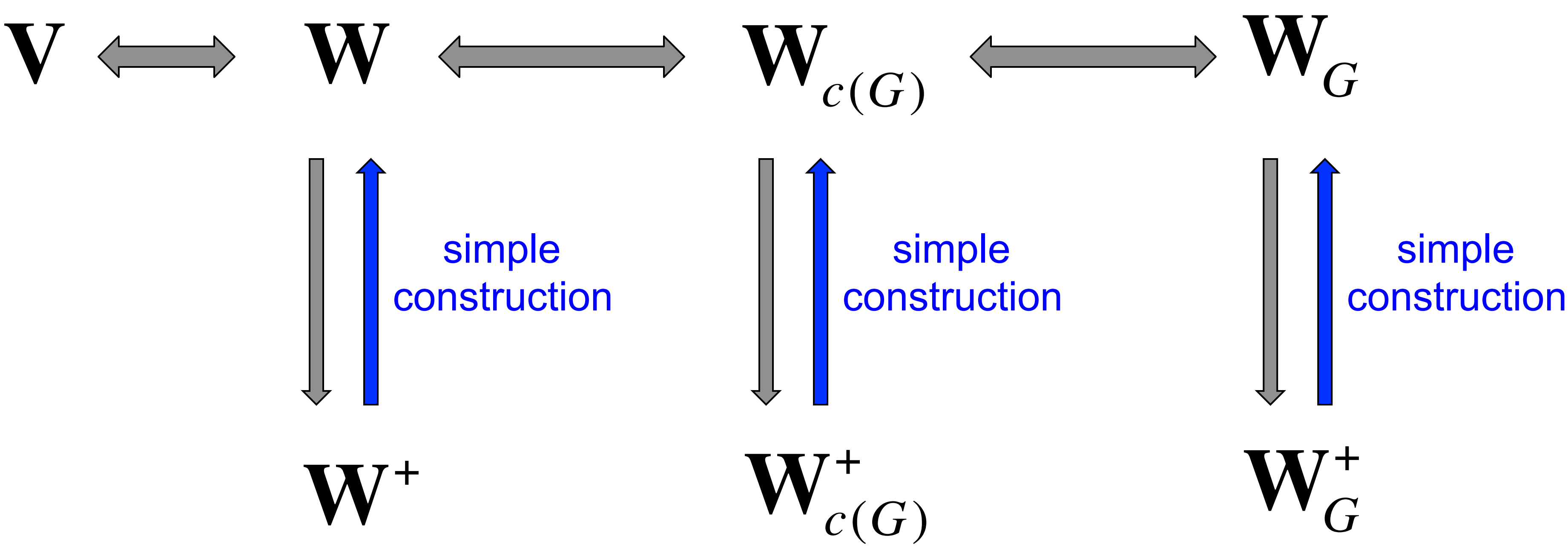
OPF-ch:

$$\min_{W_{c(G)}} C(W_G) \quad \text{subject to} \quad W_{c(G)} \in \mathbb{W}^+_{c(G)}$$

OPF-socp:

$$\min_{W_G} C(W_G) \quad \text{subject to} \quad W_G \in \mathbb{W}^+_G$$

Recap: convex relaxations



<u>SDP relaxation</u>

- tightest superset
- max # variables
- slowest

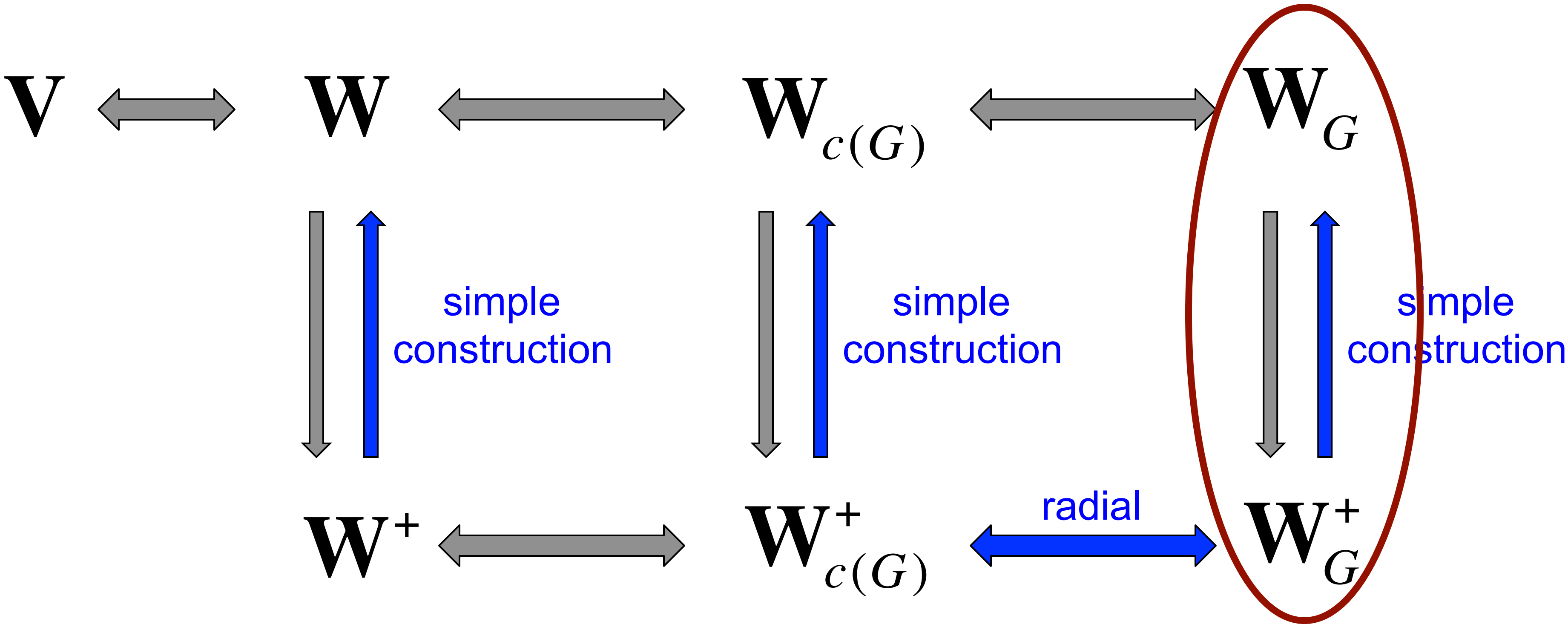
<u>Chordal relaxation</u>

- equivalent superset
- much faster for sparse networks

<u>SOCP relaxation</u>

- coarsest superset
- min # variables
- fastest

Recap: convex relaxations



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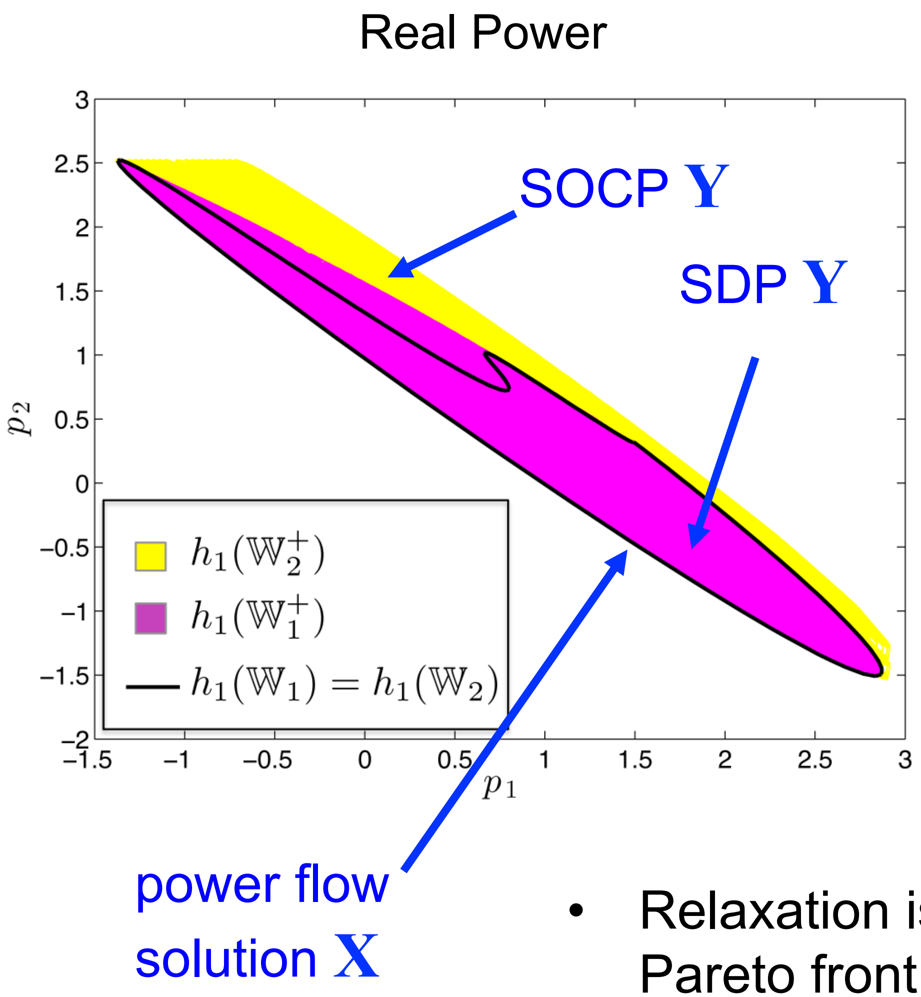
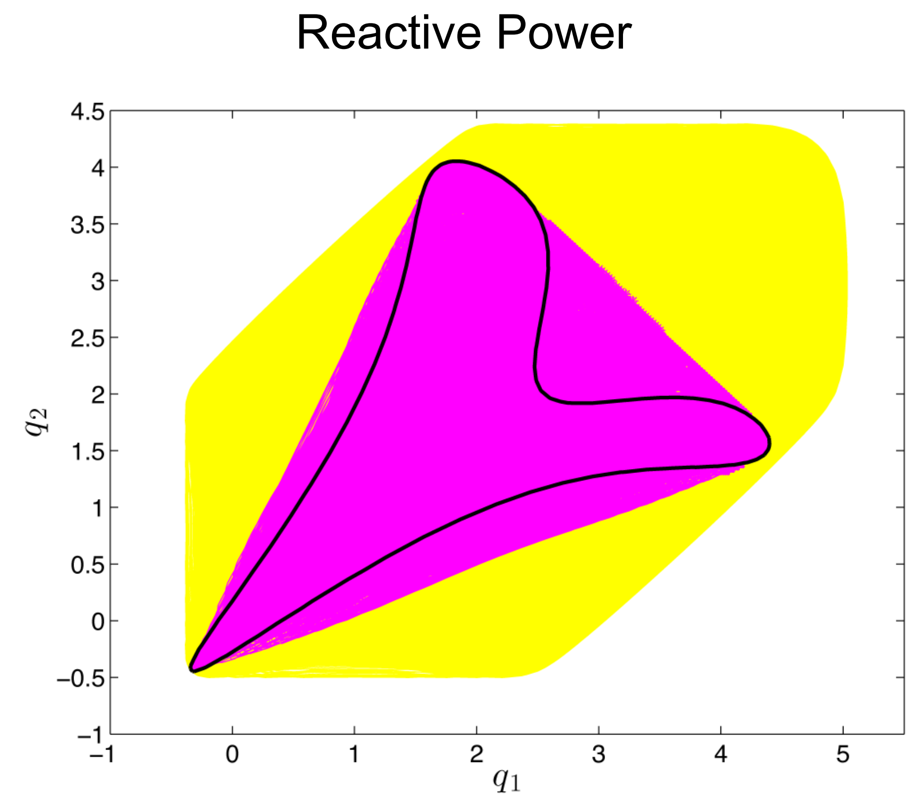
Examples





- Relaxation is exact if $\mathbf{X}$ and $\mathbf{Y}$ have same Pareto front
- SOCP is faster but coarser than SDP

Bose, Low, Teeraratkul, Hassibi TAC 2014

Without PS: SDP vs SOCP

Test case	Objective values ($/hr)		Running times (sec)	
	SDP	SOCP	SDP	SOCP
9 bus	5297.4	5297.4	0.2	0.2
14 bus	8081.7	8075.3	0.2	0.2
30 bus	574.5	573.6	0.4	0.3
39 bus	41889.1	41881.5	0.7	0.3
57 bus	41738.3	41712.0	1.3	0.3
118 bus	129668.6	129372.4	6.9	0.6
300 bus	720031.0	719006.5	109.4	1.8
2383 bus	1840270	1789500.0	-	155.3

SOCP inexact

SDP not scalable

Examples

Test case	Objective values ($/hr)		Running times (sec)		
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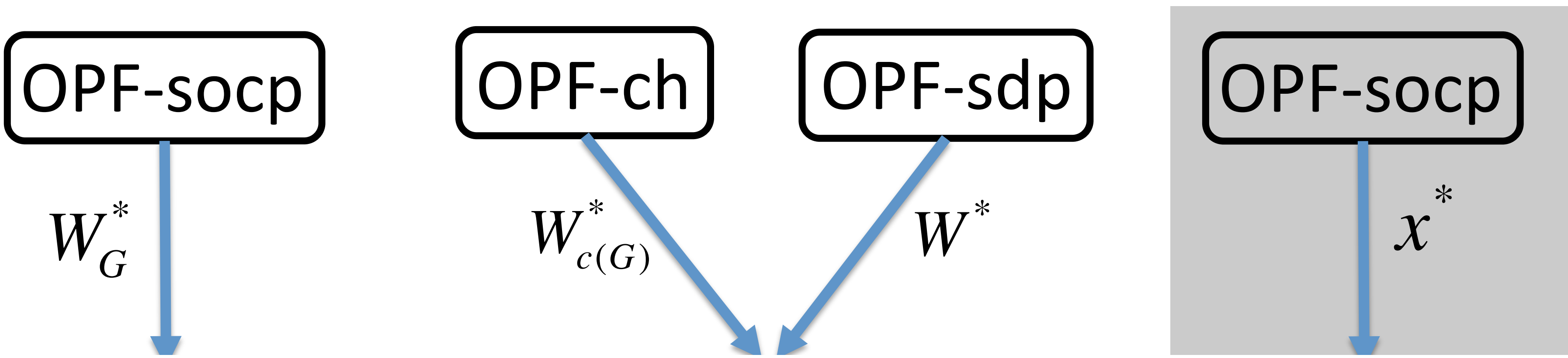
SOCP inexact

SDP not scalable



What are semidefinite relaxations of OPF?

How to check & recover global optimal ?

Branch flow model

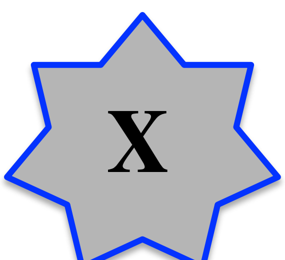
Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^* = S_{ij}$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$



SOCP relaxation

$$\sum_{j \to k} P_{jk} = \sum_{i \to j} \left(P_{ij} - r_{ij} |I_{ij}|^2 \right) + p_j$$

$$\sum_{j \to k} Q_{jk} = \sum_{i \to j} \left(Q_{ij} - x_{ij} |I_{ij}|^2 \right) + q_j$$

Branch flow model

$$\ell_{ij} := |I_{ij}|^2$$
$$v_i := |V_i|^2$$

Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

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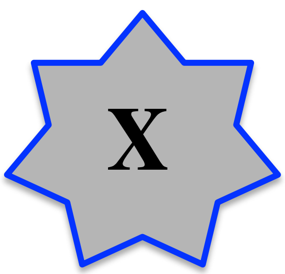

SOCP relaxation

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$

$$v_i - v_j = 2 \,\mathrm{Re}\left(z_{ij}^* S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$v_i \ell_{ij} = |S_{ij}|^2$$

$$(S, \ell, v, s) \in \mathbf{R}^{3(m+n+1)}$$

Branch flow model

$$\ell_{ij} := |I_{ij}|^2 \qquad v_i := |V_i|^2$$

Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

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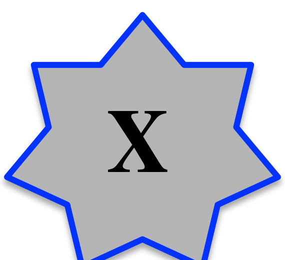


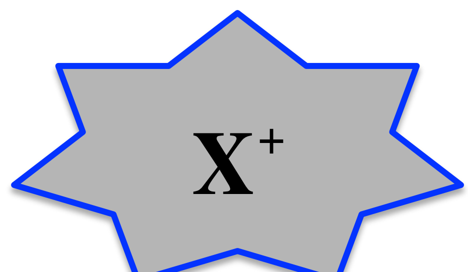
SOCP relaxation

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$

$$v_i - v_j = 2\,\mathrm{Re}\left(z_{ij}^* S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$v_i \ell_{ij} \geq |S_{ij}|^2$$

$$(S, \ell, v, s) \in \mathbf{R}^{3(m+n+1)}$$



Branch flow model

power flow solutions: $x := (S, \ell, v, s)$ satisfy

$$\sum_{j \to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j$$

$$v_i - v_j = 2\ \mathrm{Re}\left(z_{ij}^* S_{ij}\right) - \left|z_{ij}\right|^2 \ell_{ij}$$

$$\ell_{ij} v_i = \left|S_{ij}\right|^2$$

$$\ell_{ij} := \left|I_{ij}\right|^2$$

$$v_i := \left|V_i\right|^2$$

Baran and Wu 1989
for radial networks

Advantages

- Recursive structure (radial networks)
- Variables represent physical quantities
- More numerically stable

Branch flow model

$$\mathbf{X}^+ := \left\{ \begin{array}{c} x : \text{satisfies linear} \\ \text{constraints} \end{array} \right\} \cap \left\{ \ell_{jk} v_j \geq |S|^2 \right\} \quad \text{SOC}$$

$$C := \left\{ \begin{array}{l} \ell_{jk} v_j = |S|^2 \\ \text{cycle cond on } x \end{array} \right\}$$

Theorem $\mathbf{X} \equiv \mathbf{X}^+ \cap C$

Cycle condition

A relaxed solution x satisfies the cycle condition if

$$\exists \theta \quad \text{s.t.} \quad B\theta = \beta(x) \qquad \text{mod } 2\pi$$

incidence matrix;
depends on topology

$$x := (S, \ell, v, s)$$

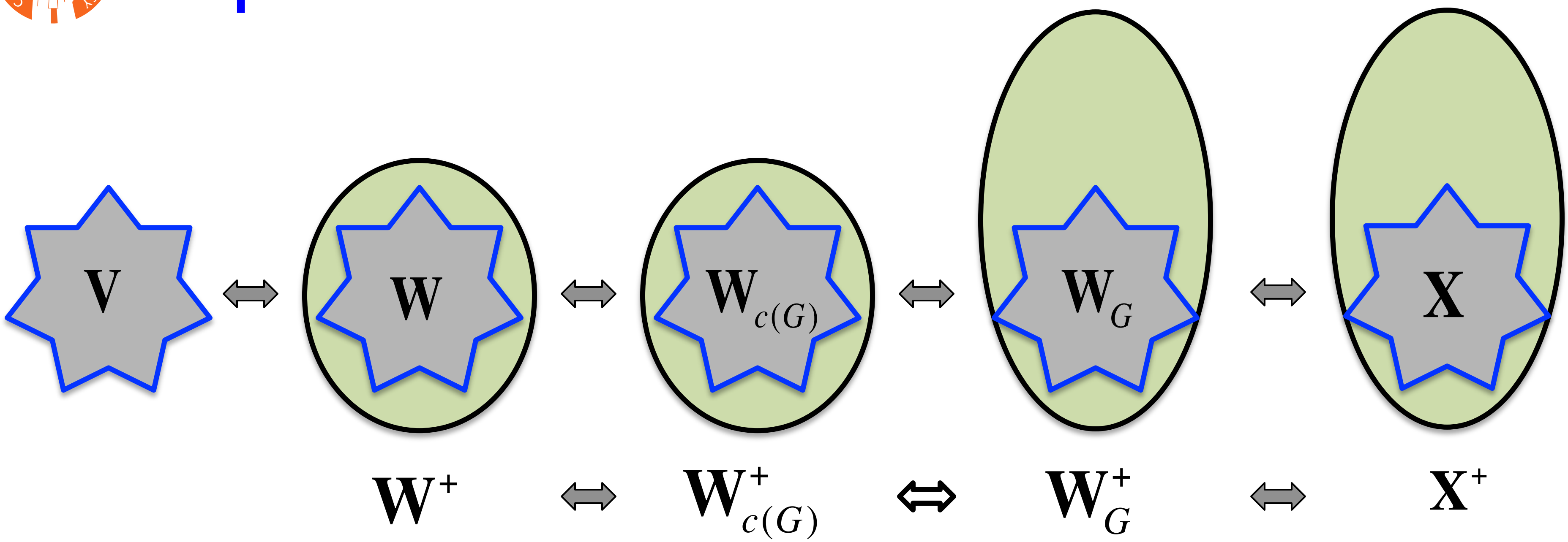
$$\beta_{jk}(x) := \angle\left(v_j - z_{jk}^H S_{jk}\right)$$

BFM: SOCP relaxation of OPF

OPF: $\min_{x \in \mathbf{X}} f(x)$

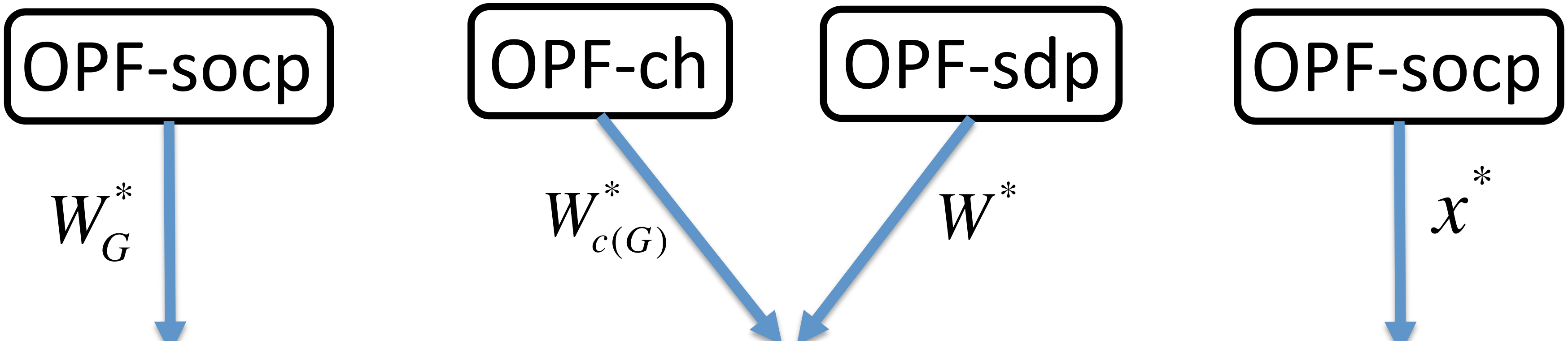
SOCP: $\min_{x \in \mathbf{X}^+} f(x)$

Equivalence

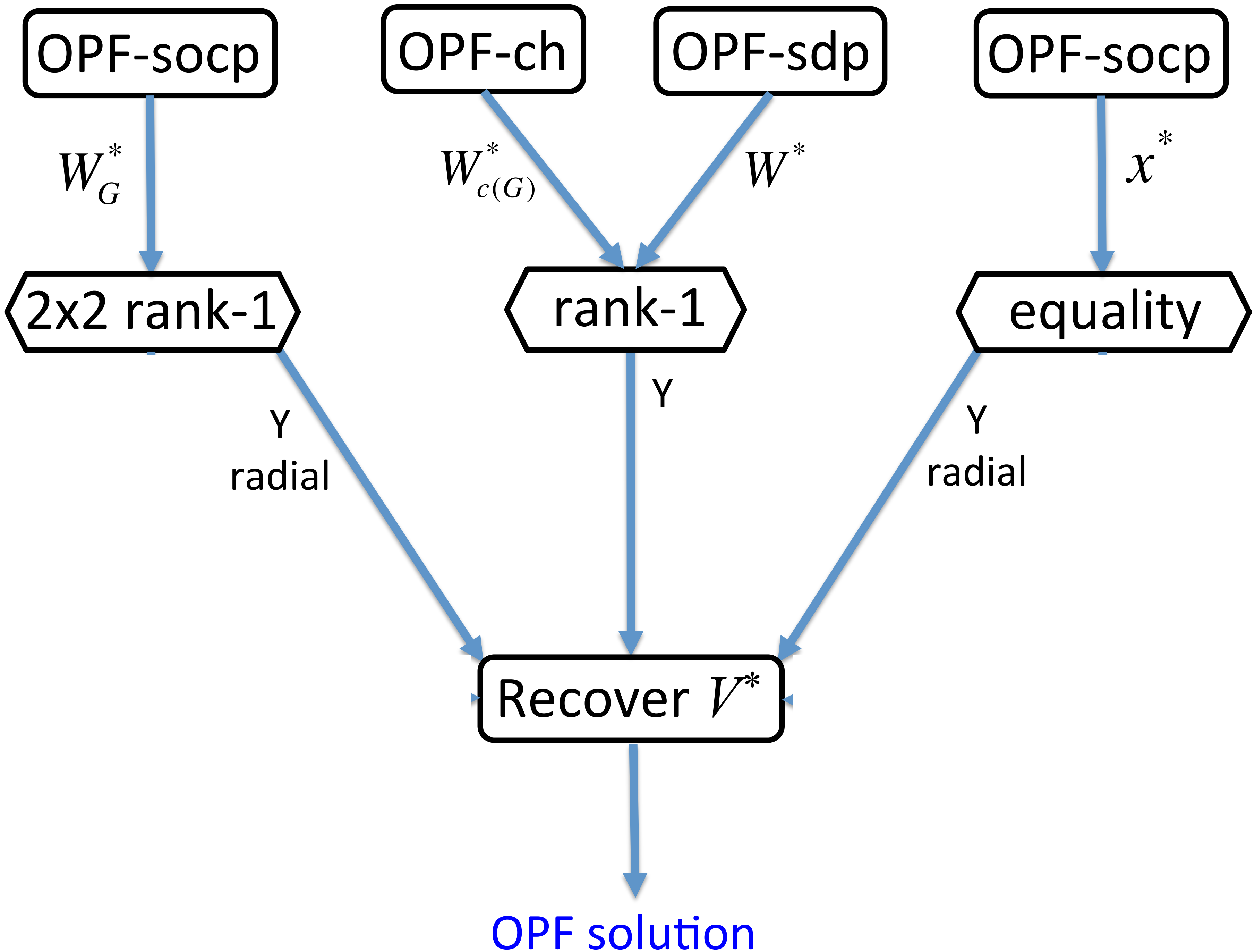


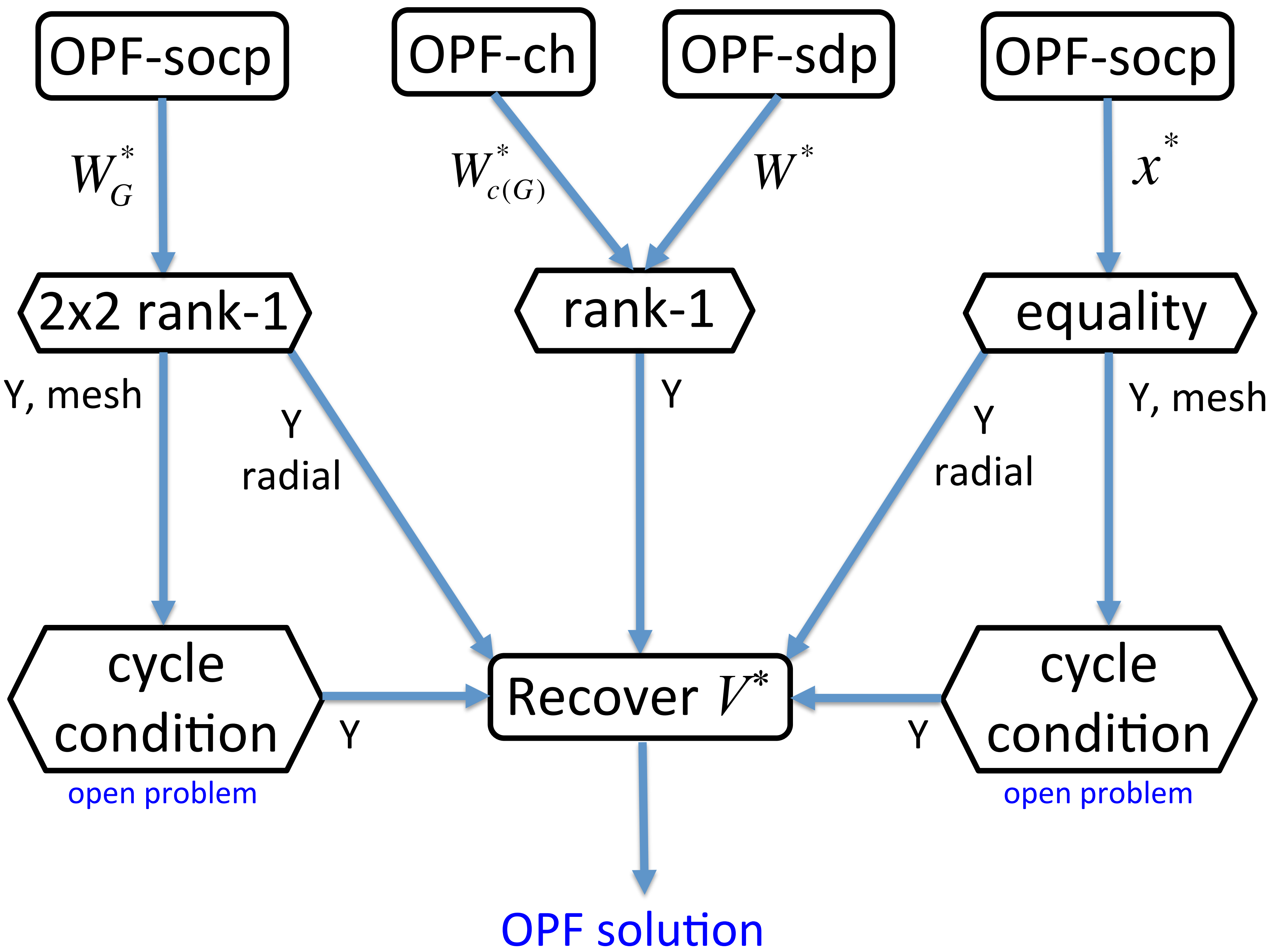
Theorem

$$\mathbf{W}_G \equiv \mathbf{X} \quad \text{and} \quad \mathbf{W}_G^+ \equiv \mathbf{X}^+$$



OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*


OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*
2x2 rank-1
rank-1
equality
Y
radial
Y
Y
radial
Recover V^*
OPF solution


OPF-socp
OPF-ch
OPF-sdp
OPF-socp
W_G^*
$W_{c(G)}^*$
W^*
x^*
2x2 rank-1
rank-1
equality
Y, mesh
Y
radial
Y
Y
radial
Y, mesh
cycle condition
Recover V^*
cycle condition
Y
Y
open problem
open problem
OPF solution

Outline

Optimal power flow (OPF)

- bus injection model, branch flow model

3 convex relaxations

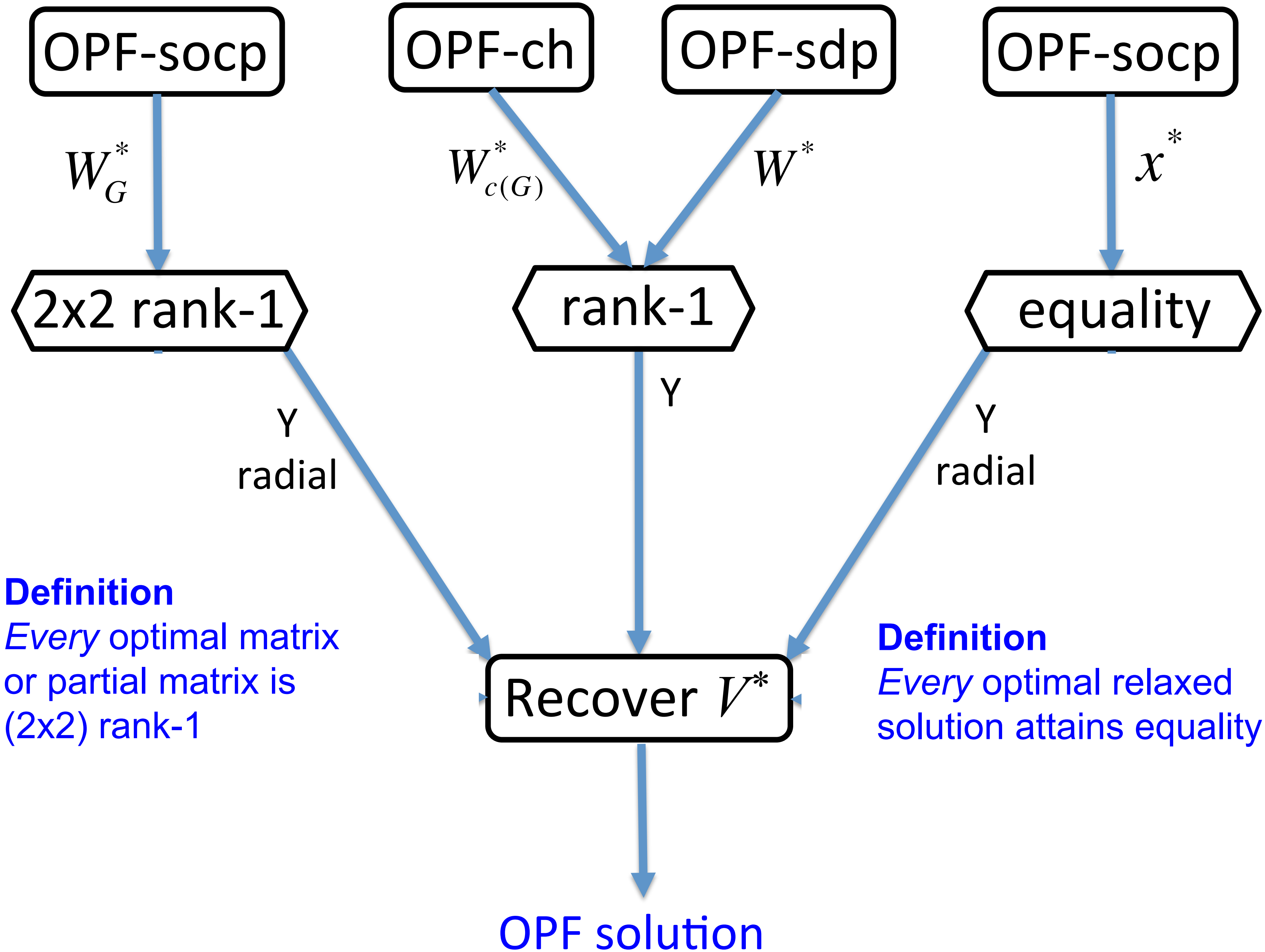
- SDP, chordal, second-order cone (SOCP)
- Relation among them

Sufficient conditions for exact relaxation

- Radial: 2/3 main conditions
- Mesh: phase shifters

Exact relaxation

A relaxation is exact if an optimal solution of the original OPF can be recovered from *every* optimal solution of the relaxation


OPF-socp
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W_G^*
$W_{c(G)}^*$
W^*
x^*
2x2 rank-1
rank-1
equality
Y
radial
Y
Y
radial
Recover V^*
OPF solution
Definition
Every optimal matrix or partial matrix is (2x2) rank-1
Definition
Every optimal relaxed solution attains equality

Summary of sufficient conds

type	condition	model	reference	remark
A	power injections	BIM, BFM	[25], [26], [27], [28], [29] [30], [16], [17]	
B	voltage magnitudes	BFM	[31], [32], [33], [34]	allows general injection region
C	voltage angles	BIM	[35], [36]	makes use of branch power flows

TABLE I: Sufficient conditions for radial (tree) networks.

network	condition	reference	remark
with phase shifters	type A, B, C	[17, Part II], [37]	equivalent to radial networks
direct current	type A	[17, Part I], [19], [38]	assumes nonnegative voltages
	type B	[39], [40]	assumes nonnegative voltages

TABLE II: Sufficient conditions for mesh networks

1. QCQP over tree

QCQP (C, C_k)

$$\begin{array}{lll} \min & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \ \leq \ b_k & k \in K \end{array}$$

graph of QCQP

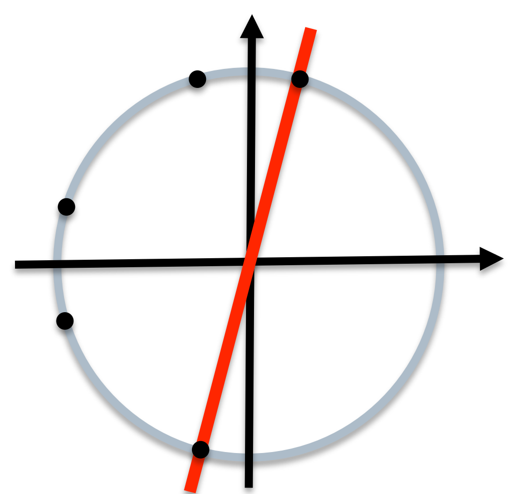
$$G(C, C_k) \ \text{ has edge } (i, j) \quad \Leftrightarrow$$

$$C_{ij} \neq 0 \ \text{ or } \ [C_k]_{ij} \neq 0 \ \text{ for some } k$$

QCQP over tree

$$G(C, C_k) \ \text{ is a tree}$$

1. Linear separability



QCQP (C, C_k)

$$\begin{array}{lll} \text{min} & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \ \leq \ b_k & k \in K \end{array}$$

Key condition

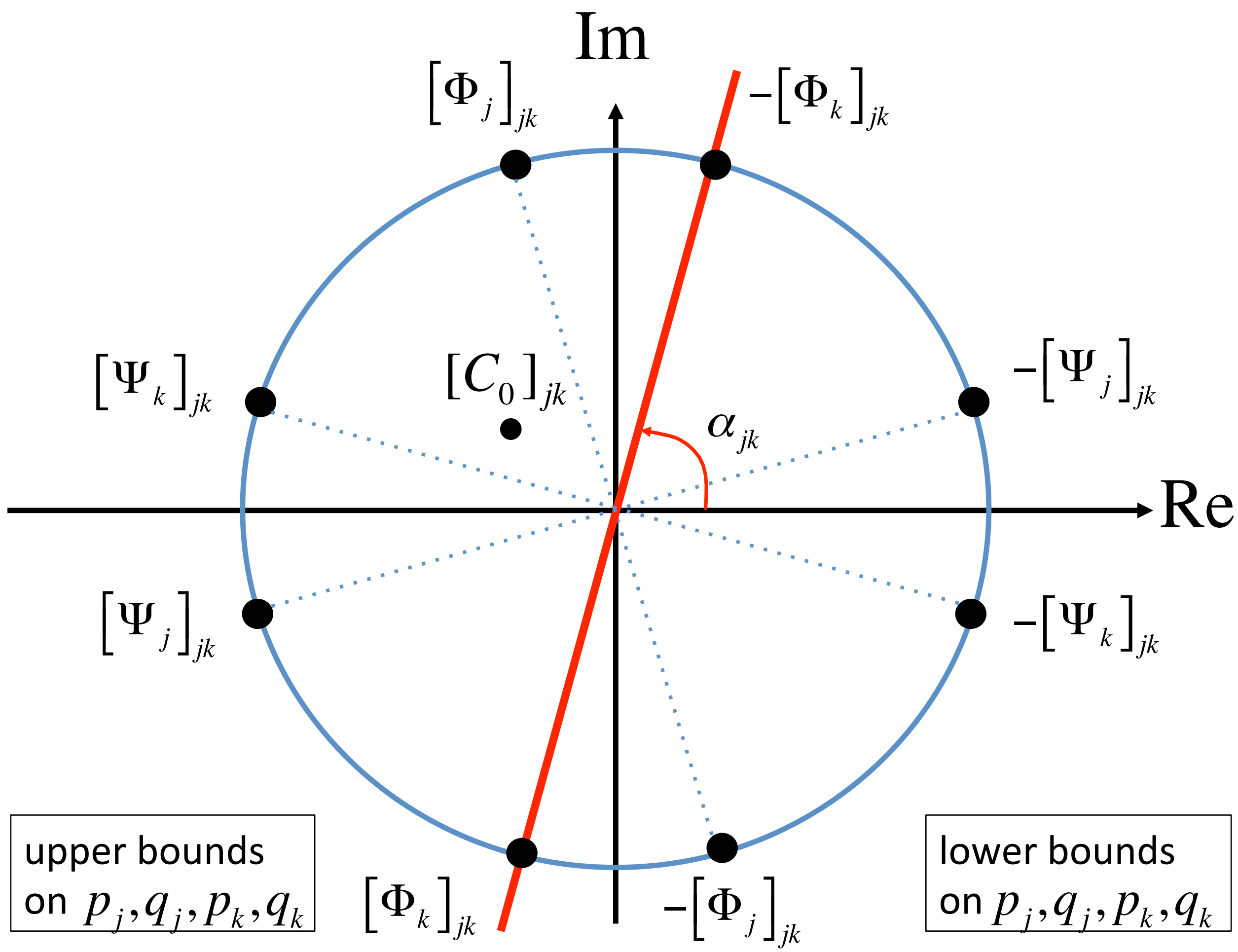
$$i \sim j: \ \left(C_{ij}, [C_k]_{ij}, \ \forall k \right) \text{ lie on half-plane through } 0$$

Theorem

SOCP relaxation is exact for QCQP over tree

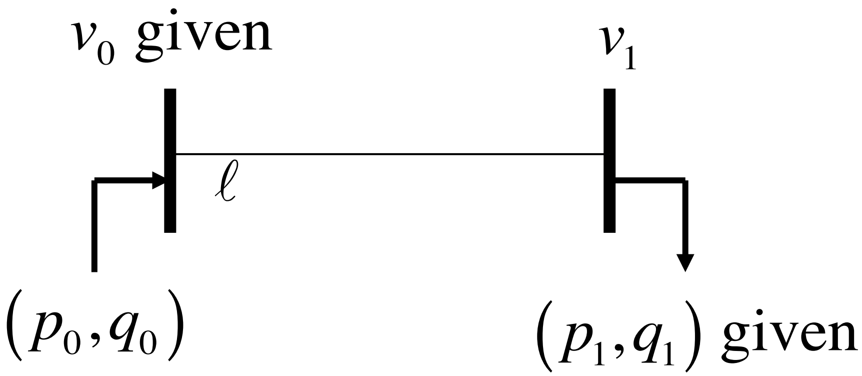
Bose et al 2012
Sojoudi, Lavaei 2013

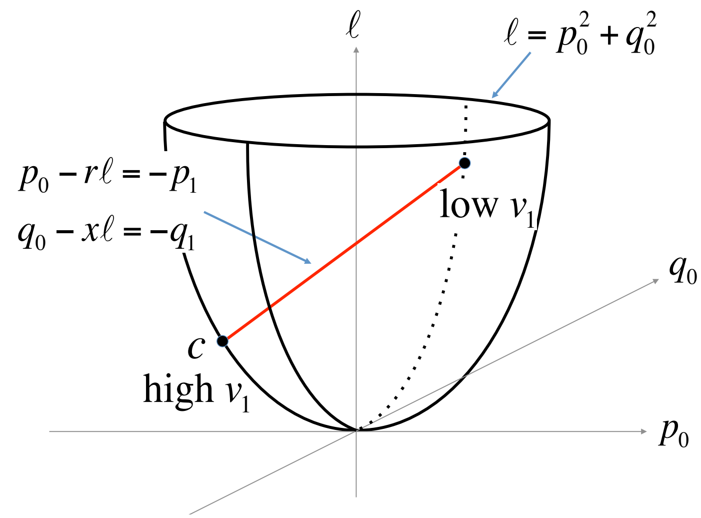
Implication on OPF



Not both lower & upper bounds on real & reactive powers at both ends of a line can be finite

2. Voltage upper bounds

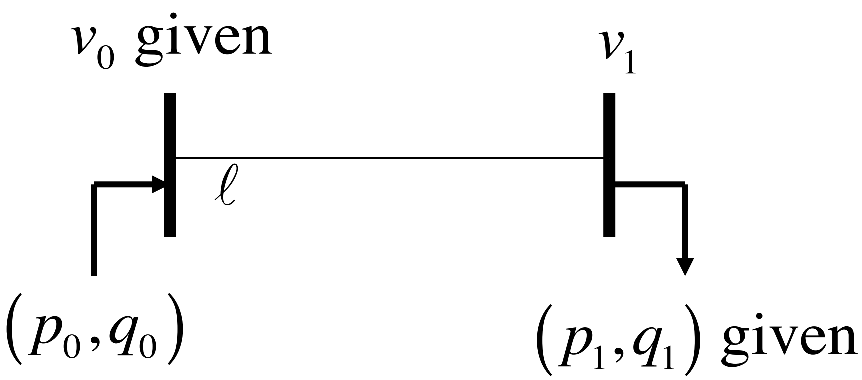


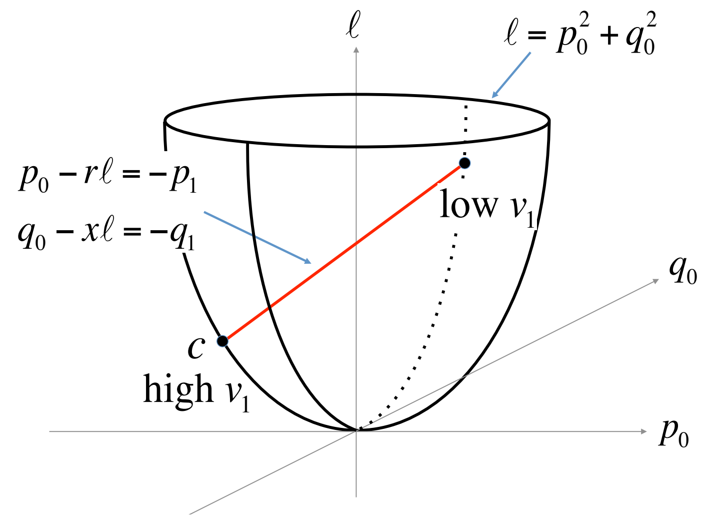


when there is no voltage constraint

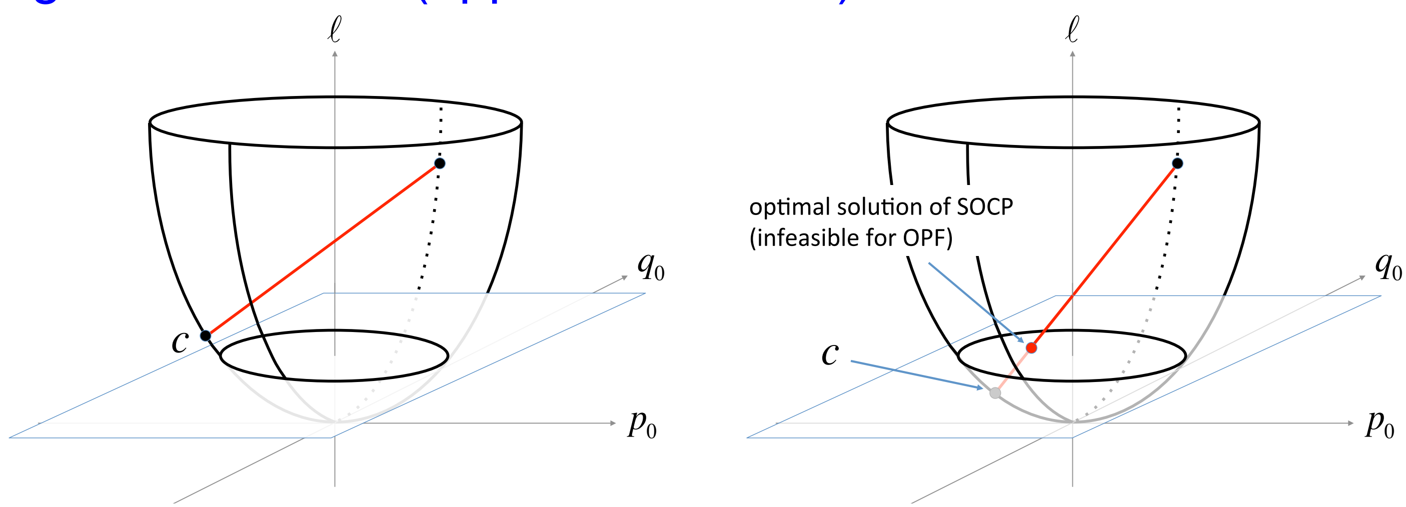
- feasible set : 2 intersection pts
- relaxation: line segment
- exact relaxation: c is optimal

2. Voltage upper bounds





voltage lower bound (upper bound on l) does not affect relaxation



(a) Voltage constraint not binding (b) Voltage constraint binding

2. Voltage upper bounds

OPF: $$\min_{x \in \mathbf{X}} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \Sigma$$

SOCP: $$\min_{x \in \mathbf{X}^+} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \Sigma$$

Key condition:

- $L(s) \le \overline{v}$ — voltages if network were lossless
- Jacobian condition — if upward current were reduced then all subsequent powers dec

  $\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0 \text{ for all } 1 \le t \le t' < k$

Theorem

SOCP relaxation is exact for radial networks

Gan, Li, Topcu, Low TAC2014

2. Voltage upper bounds

OPF: $\min_{x \in \mathbf{X}} f(x) \quad \text{s.t.} \quad \underline{v} \leq v \leq \overline{v}, \quad s \in \Sigma$

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Key condition:

- $L(s) \leq \overline{v}$
- Jacobian condition

$\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0 \text{ for all } 1 \leq t \leq t' < k$

satisfied with large margin in IEEE circuits and SCE circuits

Theorem

SOCP relaxation is exact for radial networks

Gan, Li, Topcu, Low TAC2014