Resilient distributed control of dynamical flow networks

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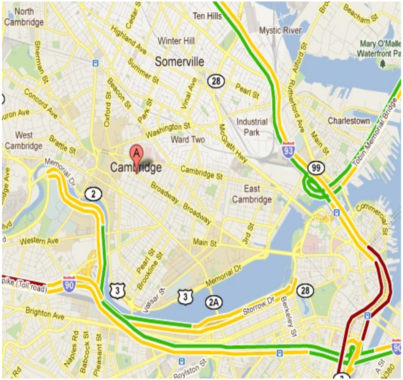
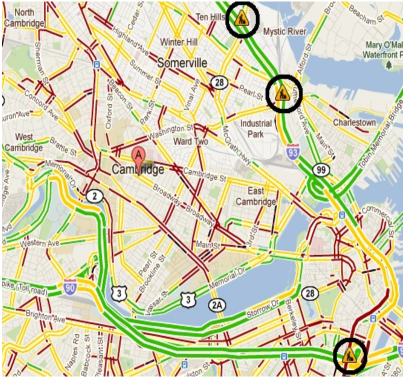
LCCC Workshop on Dynamics and Control in Networks
October 17, 2014

Large-scale infrastructure networks



typical Monday at 18:30

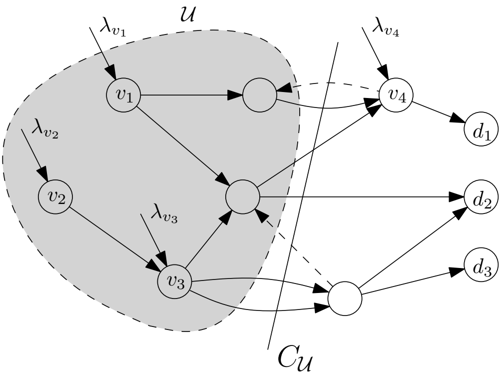


Monday, July 11, at 18:30

- good in business as usual, prone to disruptions
- cascade effects

$\Longrightarrow$ network vulnerability $\gg \sum$ component vulnerabilities

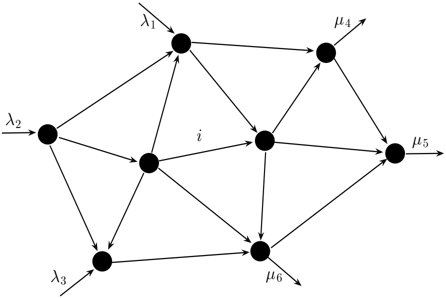
Max-flow min-cut theorem ('56)



$\exists$ feasible equilibrium flow $\quad\Longleftrightarrow\quad$ $\min_{\mathcal{U}} \{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\} > 0$

- static, centralized, global information

Optimal network flow

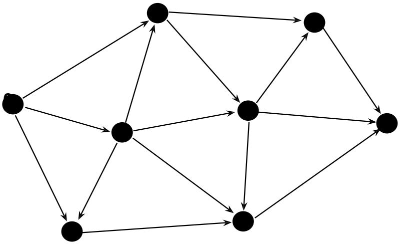


$$\min \sum_{i,j} c_{ij}(f_{ij})$$

$$\lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i = 0$$

- static, convex, rich duality theory

Wardrop equilibrium ('56)



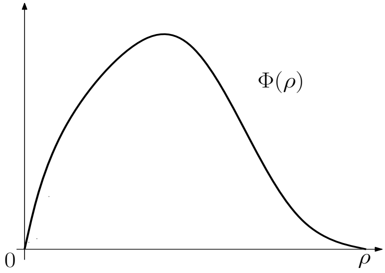
- 'the journey times in all routes actually used are equal and less than those which would be experienced on any unused route'

$$f^W \Leftrightarrow \text{Nash equilibrium of congestion game}$$

- user optimum vs social optimum: price of anarchy
- using tolls on links allows one to align f^W with any desired f^*

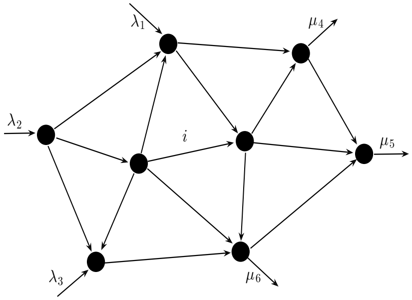
Lighthill-Whitham-Richards traffic model ('55)

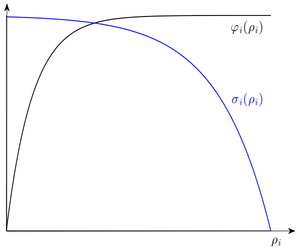
$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\Phi(\rho) = 0$$



fundamental diagram

Dynamical flow networks

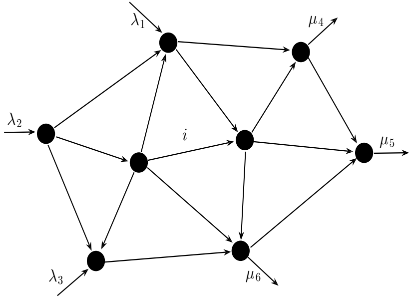


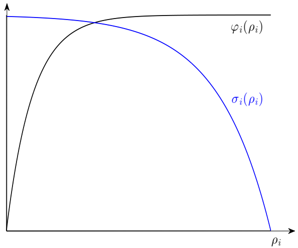


$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(\rho_i) \qquad \sum_j f_{ji} \leq \sigma_i(\rho_i)$$

Dynamical flow networks





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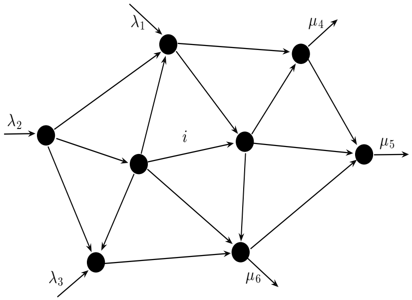
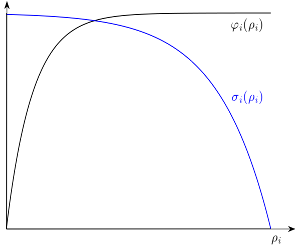
$$\mu_i + \sum_j f_{ij} \leq \varphi_i(\rho_i) \qquad\qquad \sum_j f_{ji} \leq \sigma_i(\rho_i)$$

- Daganzo'92, Gomes&Horowitz'06, Pisarski&CanudasDeWit'12, Varaiya'08, Coogan&Arcak'14, Karafyllis&Papageorgiou'14, ...

Dynamical flow networks



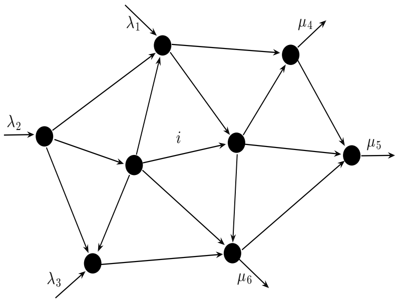


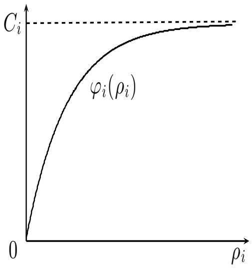
$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(\rho_i) \qquad \sum_j f_{ji} \leq \sigma_i(\rho_i)$$

- goal: scalable design of f_{ij} with good performance and resilience

Resilient decentralized routing in dynamical flow networks

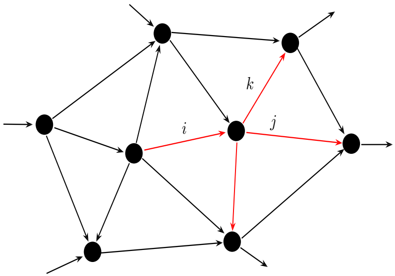


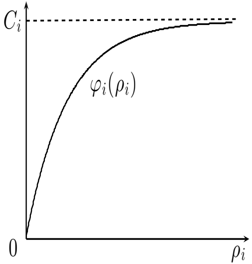


$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(\rho_i)$$

Decentralized routing





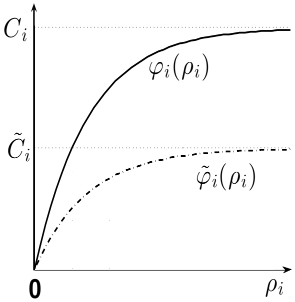
$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

flow f_{ij} depends only on local information ρ^i

$$f_{ij} = \varphi_i(\rho_i) \quad R_{ij}(\rho^i)$$

flow from i to j — demand on i — fraction routed to j

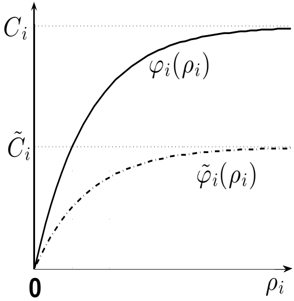
Resilience



- magnitude of perturbation $$\delta := \sum_i |\tilde{\lambda}_i - \lambda_i| + \sum_i |\tilde{C}_i - C_i|$$
- throughput loss $$\gamma := \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \sum_i (\tilde{\lambda}_i(s) - \tilde{\mu}_i(s)) \mathrm{d}s$$

$$\dot{\rho}_i = \tilde{\lambda}_i + \sum_j \tilde{f}_{ji} - \sum_j \tilde{f}_{ij} - \tilde{\mu}_i \qquad \tilde{f}_{ij} = \tilde{\varphi}_i(\rho_i) R_{ij}(\rho^i)$$

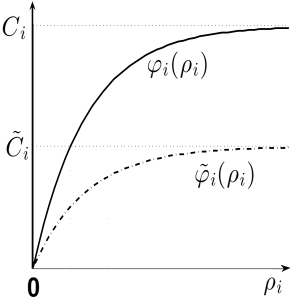
Resilience



margin of resilience $:= \inf \{\delta : \gamma > 0\}$

- $\delta :=$ magnitude of perturbation
- $\gamma :=$ throughput loss

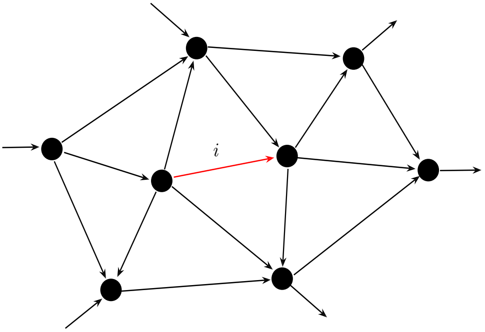
Resilience



margin of resilience $\quad \nu := \inf\{\delta : \gamma > 0\}$

- necessarily: $\nu \leq \min_{\mathcal{U}} \{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\}$
- Problem: max resilience ν over decentralized routing policies R

Resilience with fixed routing



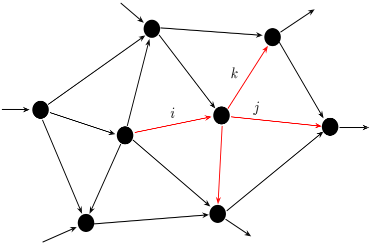
- $f_{ij} = R_{ij}\varphi_i(\rho_i)$ with constant R_{ij}
- start from equilibrium $f_i^* = \varphi_i(\rho_i^*)$

$$\nu = \min_i \{\underbrace{C_i - f_i^*}_{\text{link residual capacity}}\}$$

Resilience with locally responsive routing

(*a*) $R_{ij}(\rho^i)$ depends on local info

$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$



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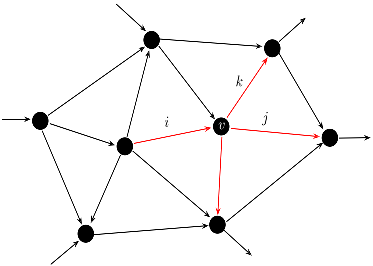

- Theorem [G.C.,K.Savla,D.Acemoglu,M.Dahleh,E.Frazzoli,TAC'13]

$$(a) \quad \Longrightarrow \quad \nu \leq \min_v \underbrace{\sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)}_{\text{node residual capacity}}$$

initial equilibrium f^*

Resilience with locally responsive routing

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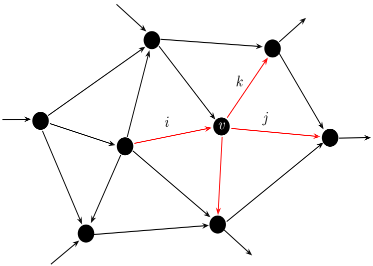

(b) $\frac{\partial}{\partial \rho_k} R_{ij} \geq 0 \qquad \forall k \neq j$

$$\rho_j \to \infty \Rightarrow \; R_{ij} \to 0$$

- Example

$$R_{ij}(\rho^i) = \frac{e^{-\beta(\rho_j + \alpha_j)}}{\sum_k e^{-\beta(\rho_k + \alpha_k)}} \qquad \beta > 0$$

Resilience with locally responsive routing

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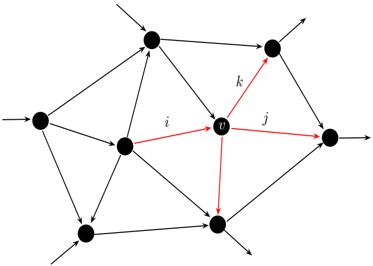

- Theorem [G.C.,K.Savla,D.Acemoglu,M.Dahleh,E.Frazzoli,TAC'13]

In acyclic networks

$$(a) \text{ and } (b) \qquad \Longrightarrow \qquad \nu = \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)$$

f^* initial equilibrium $\qquad$ f^* globally attractive

Resilience with locally responsive routing

(*a*) $R_{ij}(\rho^i)$ depends on local info

$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$

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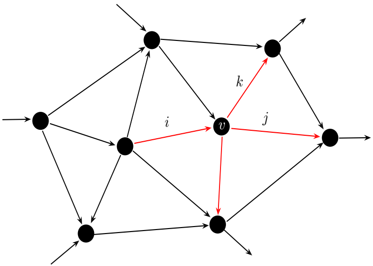

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$$(a) \text{ and } (b) \qquad \Longrightarrow \qquad \nu = \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)$$

- shocks (and information) propagate only downstream

  sub-additive effects of perturbations

Resilience with locally responsive routing

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$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$

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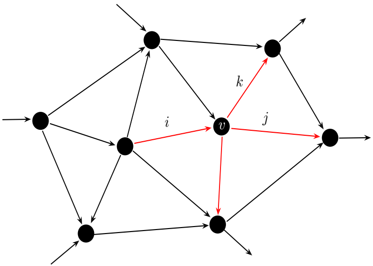
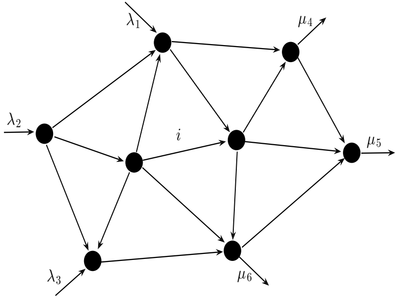


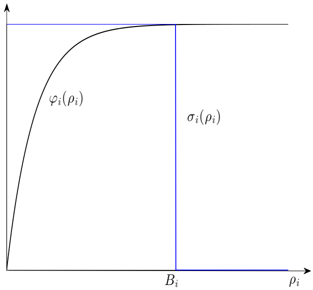
- Theorem [G.C.,K.Savla,D.Acemoglu,M.Dahleh,E.Frazzoli,TAC'13]

$$(a) \text{ and } (b) \qquad \Longrightarrow \qquad \nu = \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)$$

- typically $\qquad \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*) \ll \min_{\mathcal{U}} \{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\}$

Dynamical flow networks with cascading failures



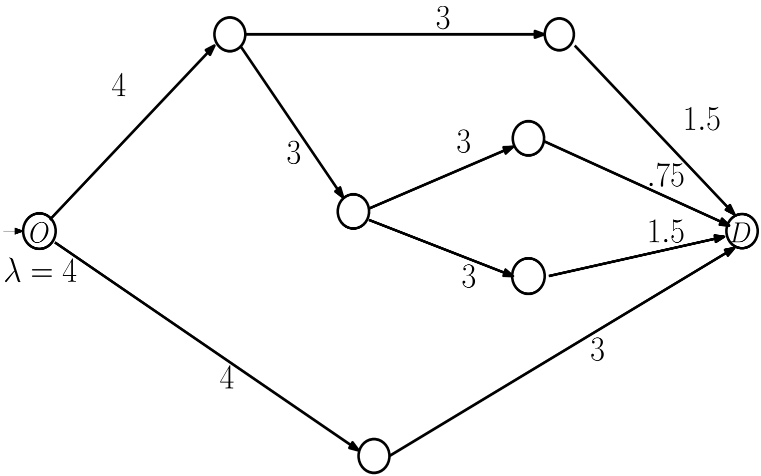


$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$f_{ij} = \varphi_i(\rho_i) R_{ij}(\rho^i)$$

- link i goes down irreversibly when $\rho_i(t) = B_i$

Dynamical flow networks with cascading failures



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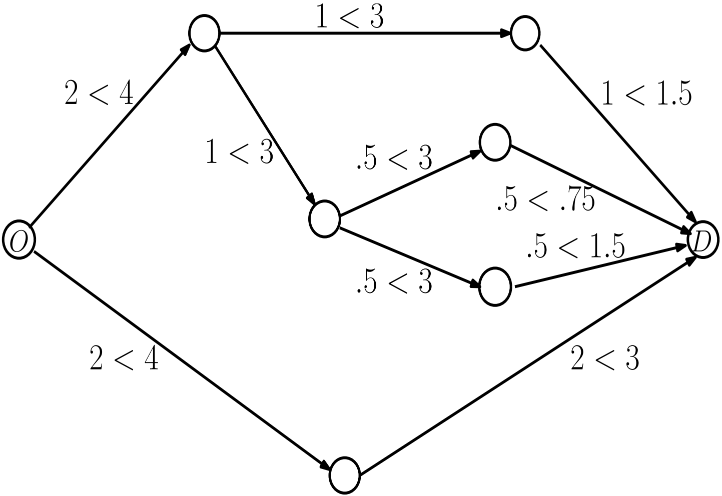

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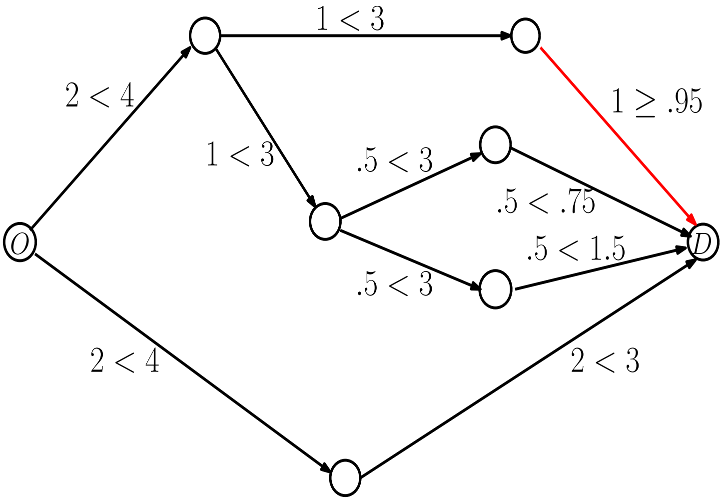

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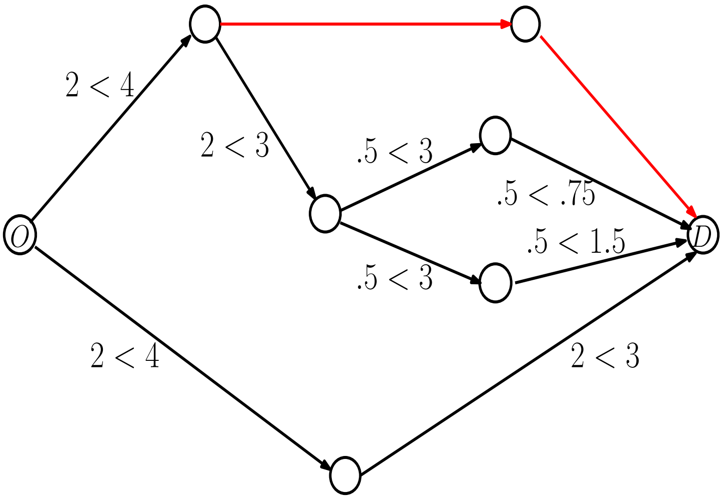

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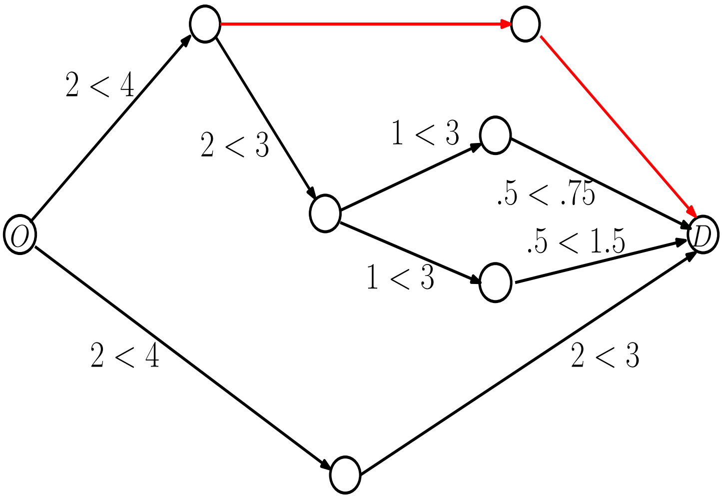

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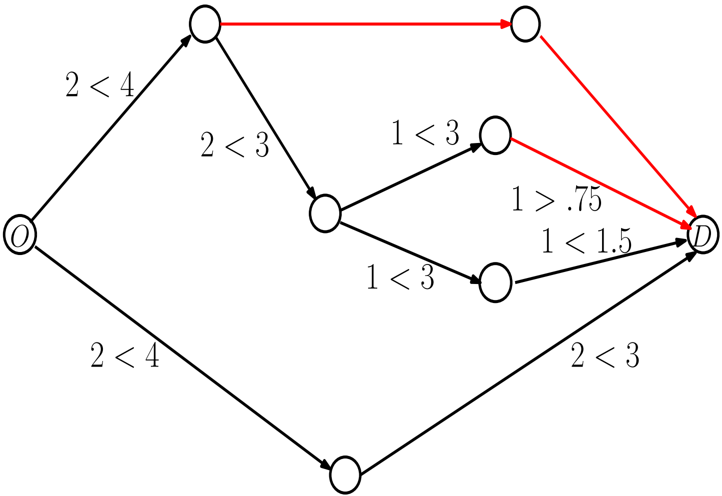

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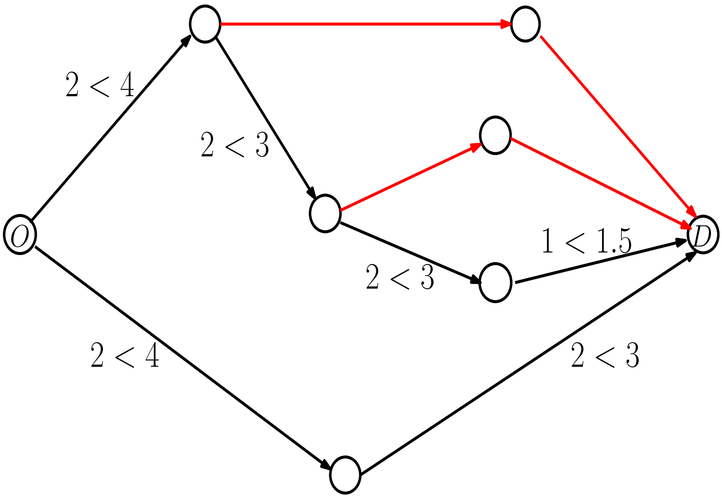

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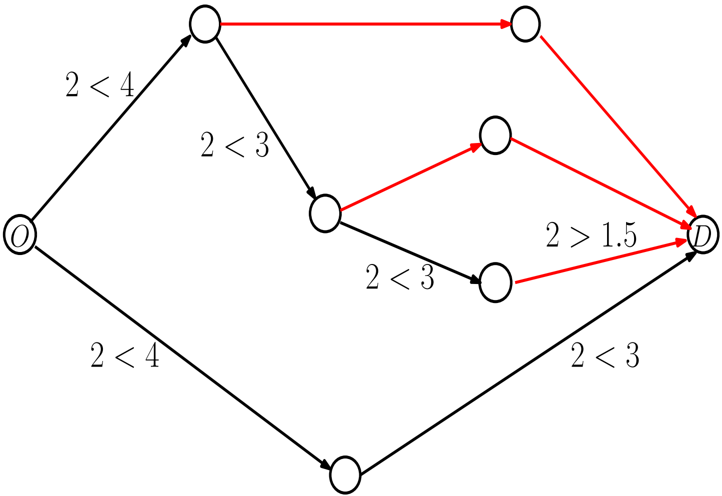

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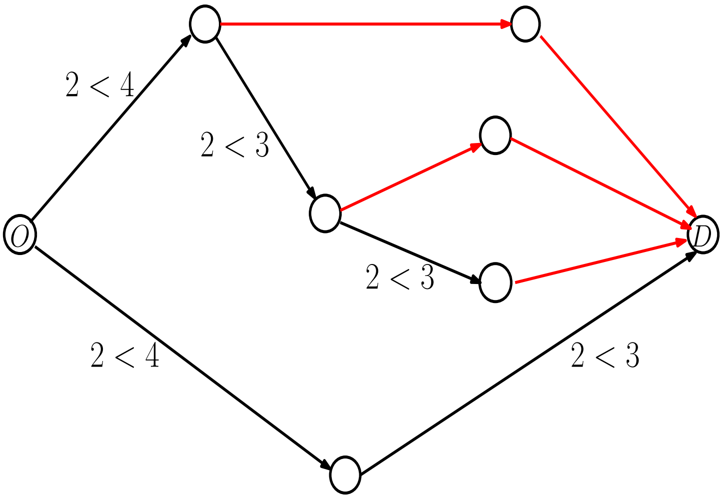

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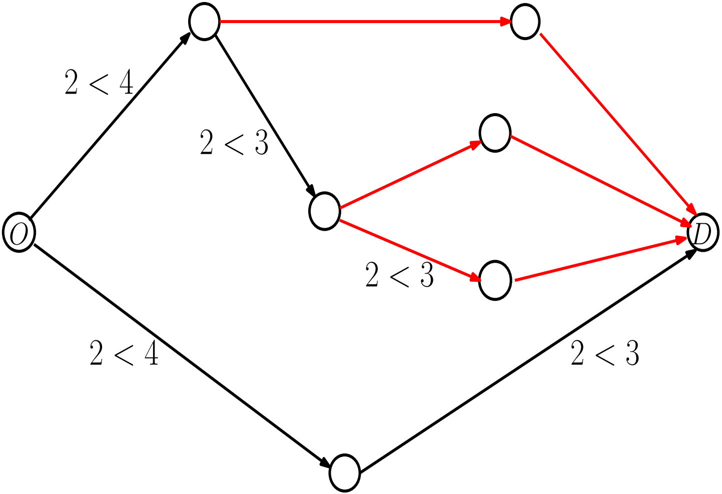

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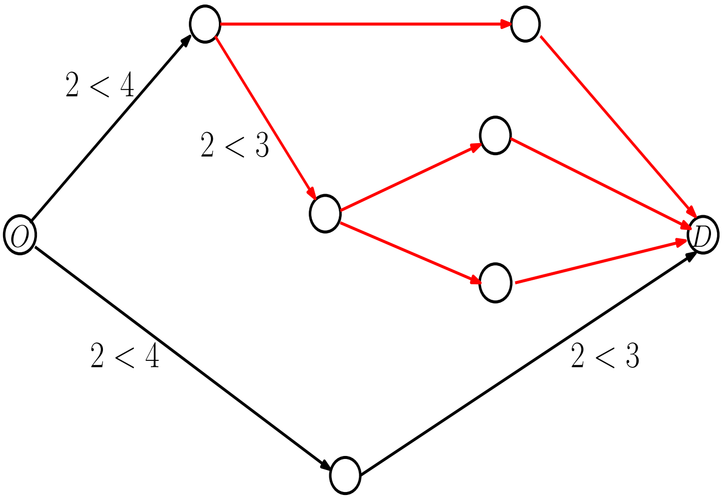

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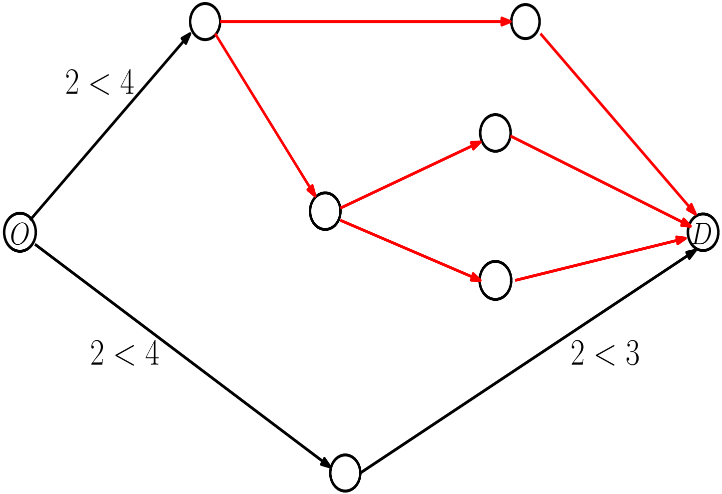

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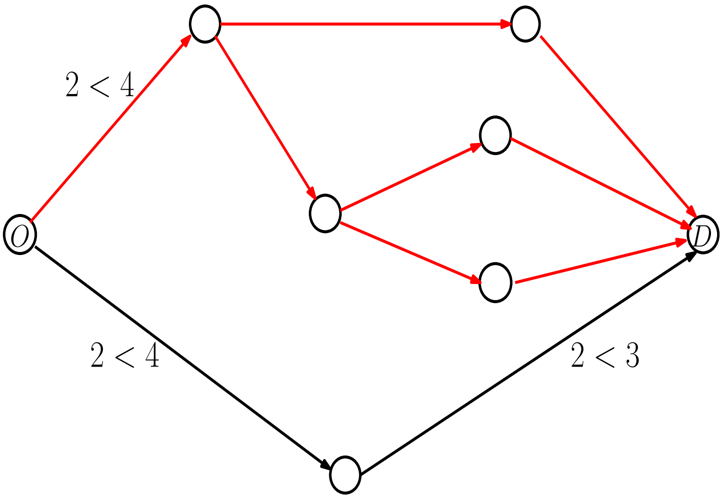

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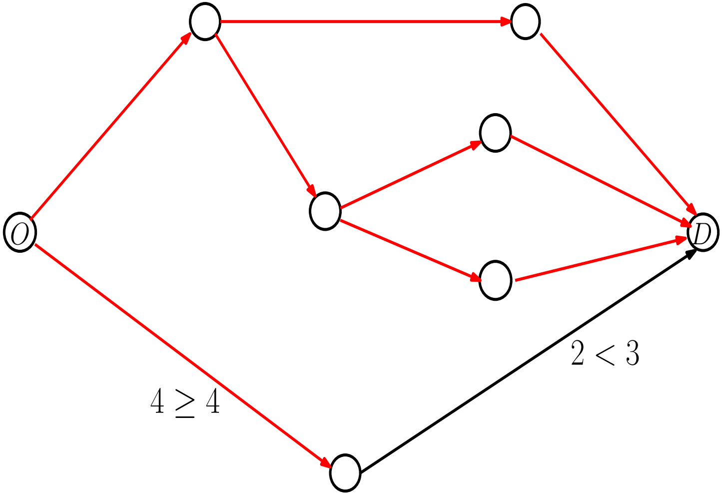

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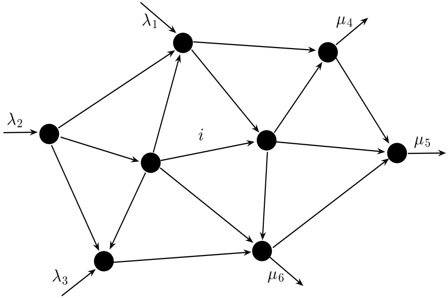
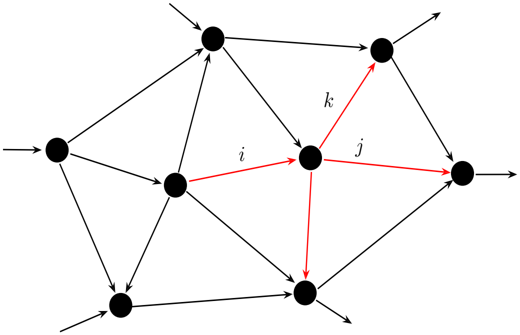


Theorem [K.Savla, G.C., M.Dahleh,'14] In acyclic networks

optimal resilience ν can be computed by dynamic programming

- shocks propagate both up- and downstream
- typically $\nu \ll \min_{\mathcal{U}} \{ C_{\mathcal{U}} - \lambda_{\mathcal{U}} \}$

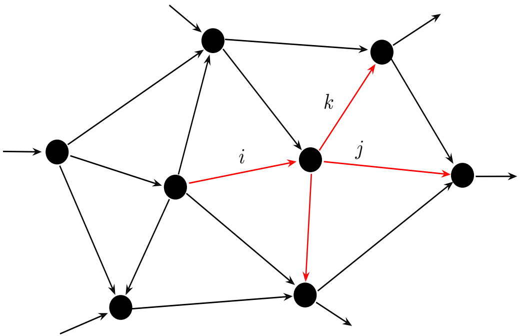
Decentralized routing with flow control



$$f_{ij} \qquad = \qquad \varphi_i(\rho_i) \qquad R_{ij}(\rho^i)$$

flow from i to j — max outflow from i — fraction routed to j

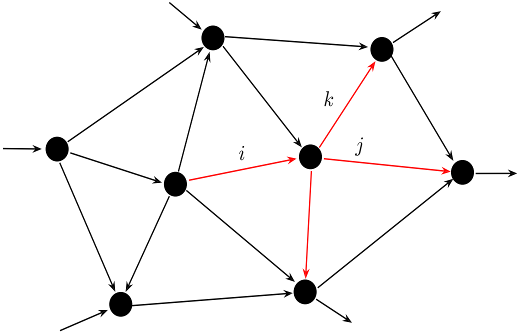
Decentralized routing with flow control



$$f_{ij} = \varphi_i(\rho_i) R_{ij}(\rho^i)$$

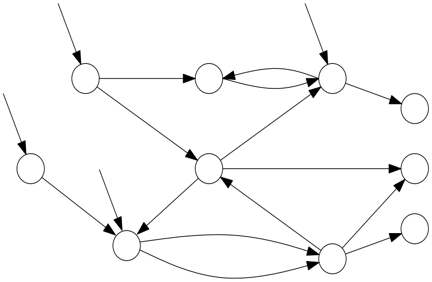
- relax $\sum_{j \in \mathcal{E}_i^+} R_{ij} = 1$ to $\sum_{j \in \mathcal{E}_i^+} R_{ij} \leq 1,$ still decentralized

Decentralized monotone routing with flow control



$$R_{ij}(\rho^i) = \frac{e^{-\beta(\rho_j + \alpha_{ij})}}{e^{-\beta(\rho_i + \alpha_{ii})} + \sum_{k \in \mathcal{E}_i^+} e^{-\beta(\rho_k + \alpha_{ik})}}$$

Decentralized monotone routing with flow control

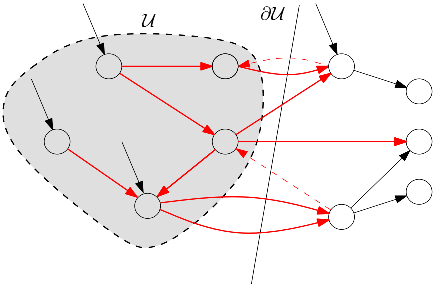


Theorem [G.C.,E.Lovisari,K.Savla,TCONES'14] :

Decentralized monotone routing, both with finite and infinite buffer

- $\min_{\mathcal{U}}\{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\} > 0 \Longrightarrow \quad \exists$ equilibrium ρ^* s.t. $\rho(t) \to \rho^* \quad \forall \rho(0)$

Decentralized monotone routing with flow control

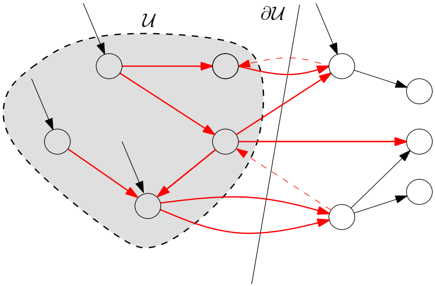


Theorem [G.C.,E.Lovisari,K.Savla,TCONES '14] :

Decentralized monotone routing, infinite buffer

- $\min_{\mathcal{U}}\{C_{\mathcal{U}}-\lambda_{\mathcal{U}}\}<0 \Longrightarrow$ minimal throughput loss (graceful degradation)

Decentralized monotone routing with flow control

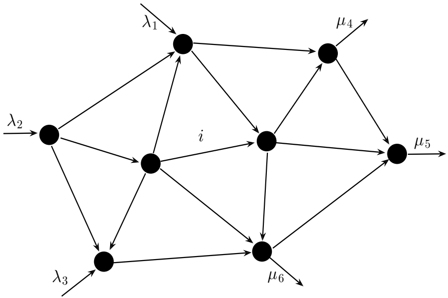


Theorem [G.C.,E.Lovisari,K.Savla,TCONES '14] :

Decentralized monotone routing, finite buffer

- $\min_{\mathcal{U}}\{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\} < 0 \Longrightarrow$ links left of cut fail simultaneously

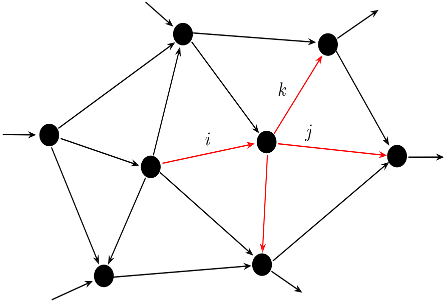
Beyond throughput and resilience



- decentralized routing + flow control achieves optimal resilience
  implicitly propagating information through the network

- for other performance measures (e.g., delay at equilibrium)
  communication / distributed optimization layer necessary

What equilibrium?



$$R_{ij}(\rho^i) = \frac{e^{-\beta(\rho_j+\alpha_{ij})}}{e^{-\beta(\rho_i+\alpha_{ii})} + \sum_{k\in\mathcal{E}_i^+} e^{-\beta(\rho_k+\alpha_{ik})}}$$

$$\alpha_{ij} = -\rho_j^* - \beta^{-1}\log(f_{ij}^*) \qquad \alpha_{ii} = -\rho_i^* - \beta^{-1}\log(\varphi_i(\rho_i^*) - \textstyle\sum_j f_{ij}^*)$$

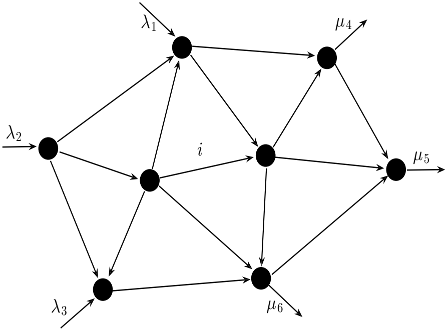
Equilibrium optimization

$$\min_{x,y} \sum_i \Psi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} = \sum_j f_{ij} + \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} \leq \sigma_i(x_i)$$



[E.Lovisari,G.C.,A.Rantzer,K.Savla,'14]

If φ_i, σ_i, and $-\psi_i$ are concave (linear):

- equilibrium optimization is convex (linear) problem

suitable for distributed solutions (ADMM)

Equilibrium optimization

$$\min_{x,y} \sum_i \Psi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} = \sum_j f_{ij} + \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(x_i)$$

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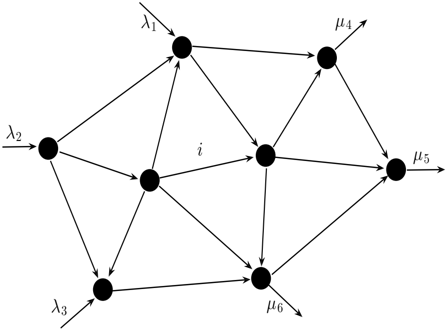


[E.Lovisari,G.C.,A.Rantzer,K.Savla,'14]

If φ_i, σ_i, and $-\psi_i$ are concave (linear):

- equilibrium optimization is convex (linear) problem

analogous problem when routing is fixed (or constrained)

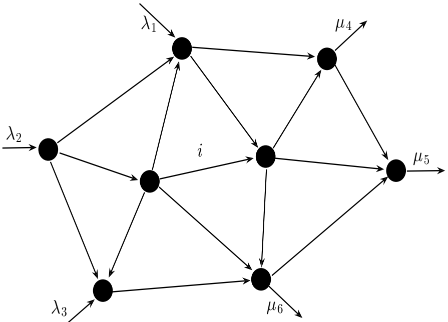
Optimal control

$$\min_{x,y} \sum_i \int_0^T \Psi_i(x_i(t))\mathrm{d}t$$

$$\dot{x}_i = \lambda_i + \sum_j (f_{ji} - f_{ij}) + \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} \leq \sigma_i(x_i)$$

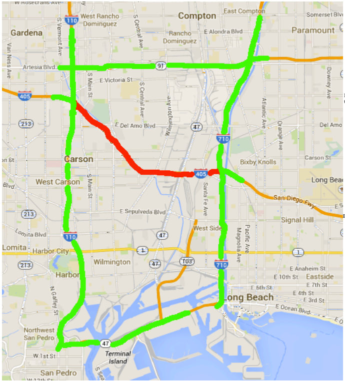


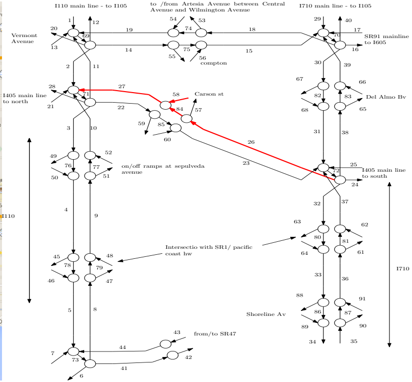
[E.Lovisari,G.C.,A.Rantzer,K.Savla,'14]

If φ_i, σ_i, and $-\psi_i$ are concave (linear):

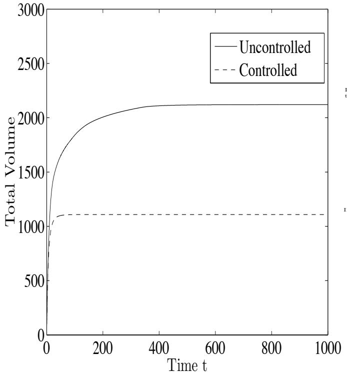
- optimal control is convex (linear) problem

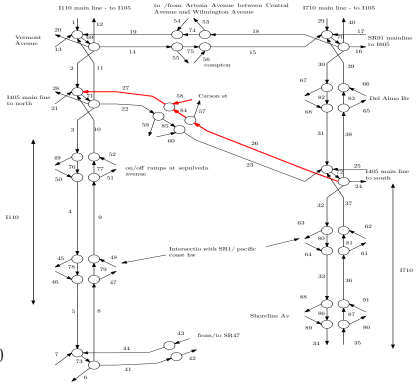
Highway net in Long Beach





Highway net in Long Beach





Conclusion

Dynamical flow networks:

- resilience and cascading failures with decentralized routing
- distributed equilibrium optimization and optimal control
- proofs: exploit monotonicity and l_1-contraction

Dynamical flow networks beyond transportation:

- production networks
- distribution networks