It may “easier to approximate” decentralized LQG problems

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Problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n] + v_1[n]$$
$$y_2[n] = x[n] + v_2[n]$$

$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1u_1^2[n] + r_2u_2^2[n]]$$

where

$$u_1[n] = f_n(y_1[0], \cdots, y_1[n])$$
$$u_2[n] = g_n(y_2[0], \cdots, y_2[n])$$

Problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$ i.i.d. Gaussian $\mathcal{N}(0, \sigma_w^2)$

$$y_1[n] = x[n] + v_1[n]$$ i.i.d. Gaussian $\mathcal{N}(0, \sigma_{v1}^2)$

$$y_2[n] = x[n] + v_2[n]$$ i.i.d. Gaussian $\mathcal{N}(0, \sigma_{v2}^2)$

$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n] + r_2 u_2^2[n]]$$

where

$$u_1[n] = f_n(y_1[0], \cdots, y_1[n])$$
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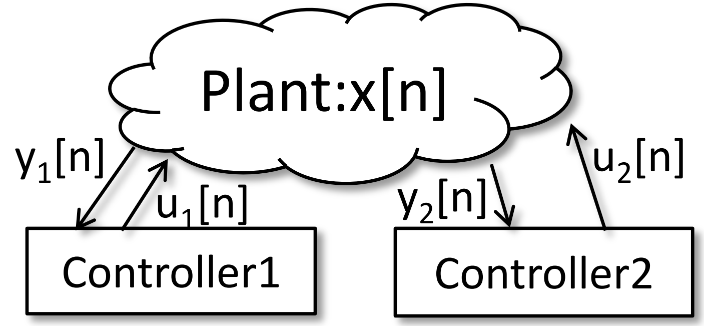


Infinite-Horizon Average-Cost Decentralized Linear Quadratic Gaussian with Scalar Plant and Two Controllers

Motivation: Automatic Driving System

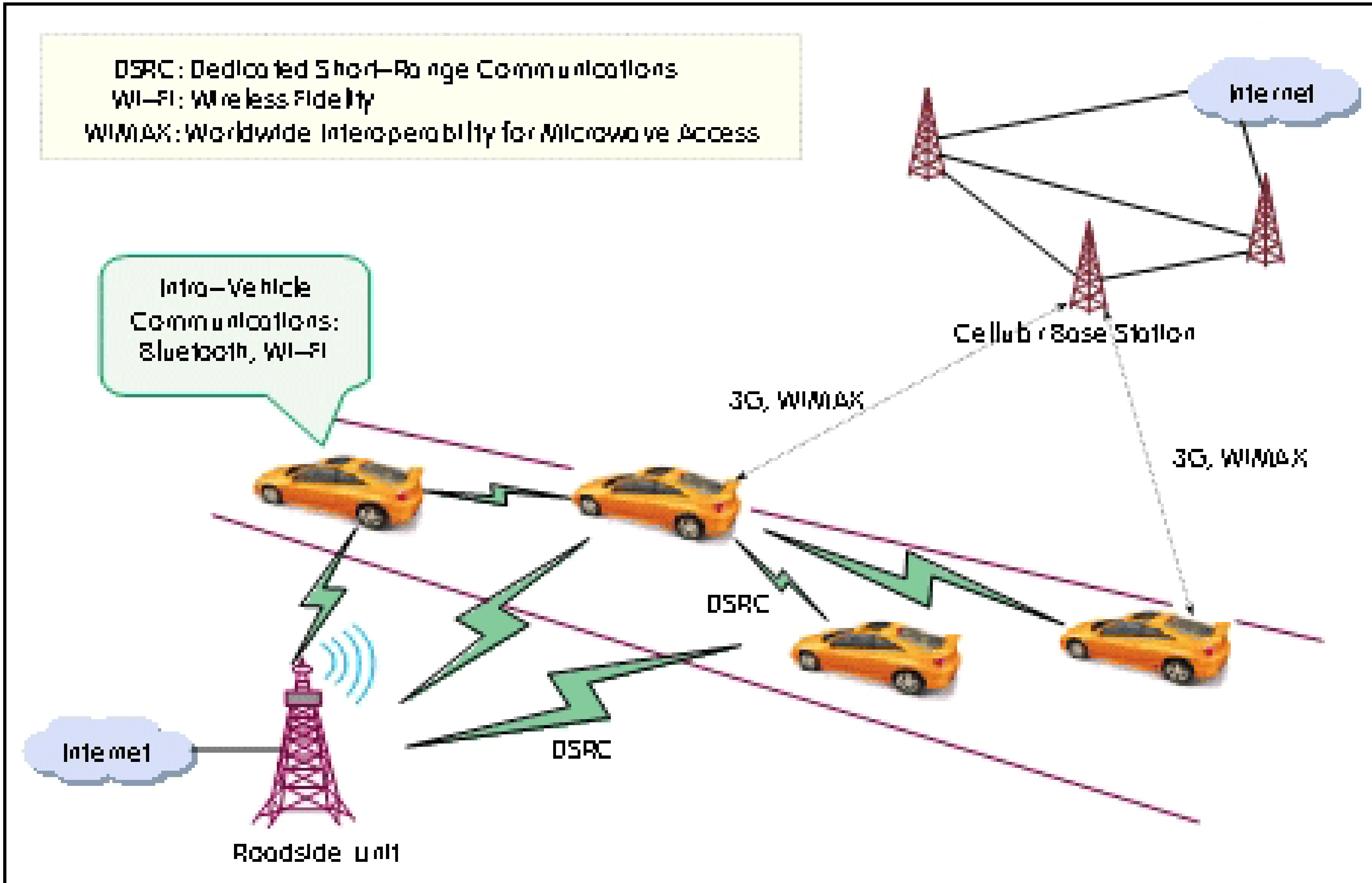
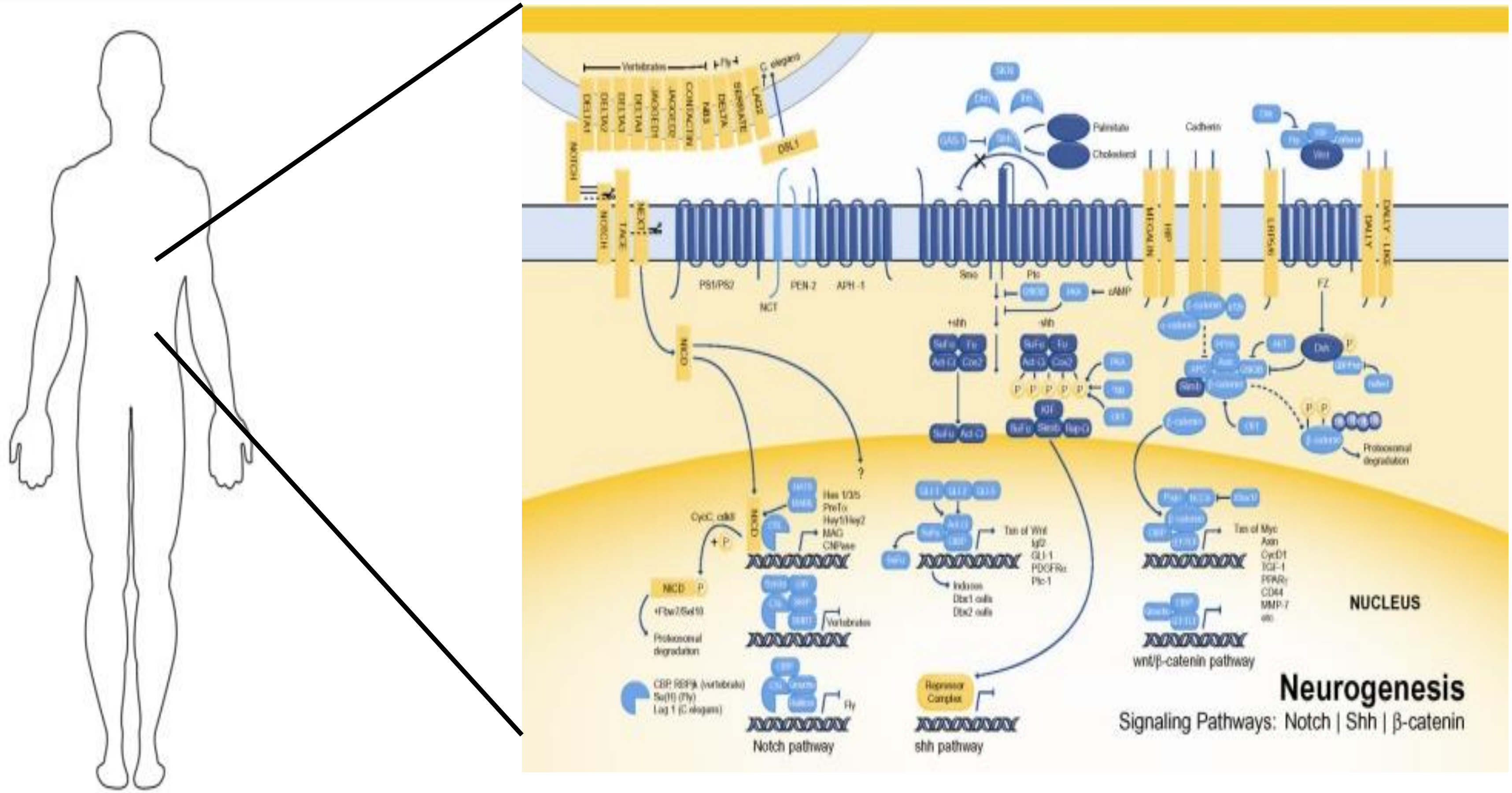


Figure 2. Wireless technologies for future vehicular communications.

Motivation: UAV (Unmanned Aerial Vehicle)

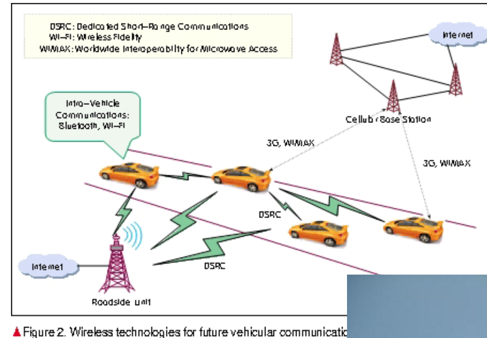
Motivation: Power Grid

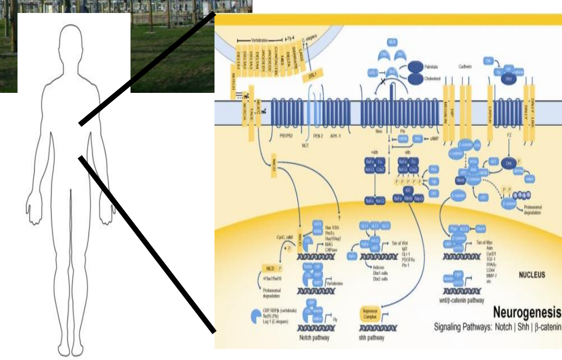
Motivation: Biological System



Traditional System vs Modern System

VS





Main Difference: Distributedness

Decentralized LQG Problem

$$x[n+1] = Ax[n] + B_1 u_1[n] + \cdots + B_m u_m[n] + w[n]$$

$$y_1[n] = C_1 x[n] + v_1[n]$$

$$\vdots$$

$$y_m[n] = C_m x[n] + v_m[n]$$

$$\inf_{u_i} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[x^*[n]Qx[n] + \sum_{1\le i\le m} u_i^*[n] R_i u_i[n]]$$

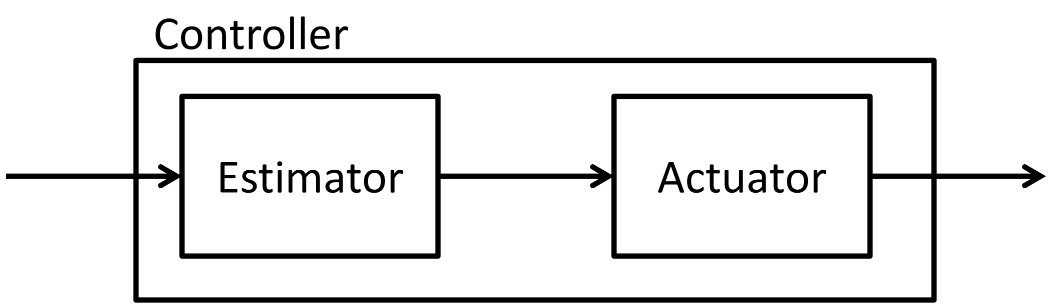
History

- Centralized LQG Problem

$$x[n+1] = Ax[n] + Bu[n] + w[n]$$
$$y[n] = Cx[n] + v[n]$$
$$\inf_{u} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \mathbb{E}[x^*[n]Qx[n] + u[n]^*Ru[n]]$$

Two Lessons

- Linear Controller is Optimal (Finite-Dimensional Solution)
- Estimation-Control Separation



History

- Centralized LQG Problem

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Two Lessons

- Linear Controller is Optimal (Finite-Dimensional Solution)
- Estimation-Control Separation

→ Linear Controller is also used for Nonlinear Systems

→ Adaptive Controllers also use Estimation-Control Structure

History

- Decentralized LQG Problem

$$x[n+1] = Ax[n] + B_1 u_1[n] + \cdots + B_m u_m[n] + w[n]$$

$$y_1[n] = C_1 x[n] + v_1[n]$$

$$\vdots$$

$$y_m[n] = C_m x[n] + v_m[n]$$

$$\inf_{u_i} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \mathbb{E}[x^*[n]Qx[n] + \sum_{1\leq i\leq m} u_i^*[n]R_i u_i[n]]$$

Negative Results

- Linear Controller is Not Optimal
- Estimation-Control Separation does Not hold

History

- Decentralized LQG Problem

$$x[n+1] = Ax[n] + B_1 u_1[n] + \cdots + B_m u_m[n] + w[n]$$

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As Optimization Problem

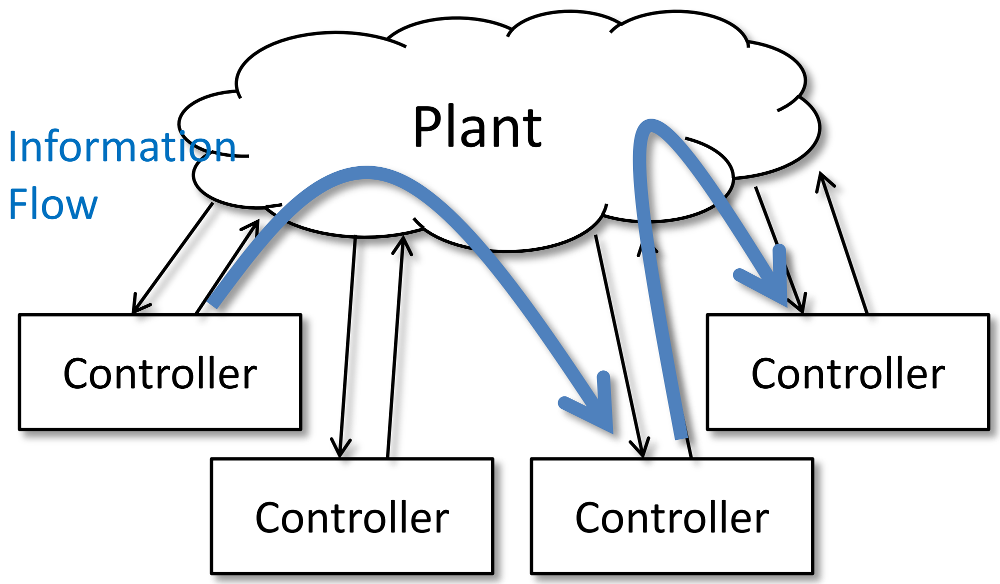
- Infinite Dimensional Optimization problem
- Non-convex problem
- Curse of Dimensionality from Dynamic Program

History

- Decentralized LQG Problem

Why it is hard?

- Implicit Communication between Controllers



History

Decentralized LQG Problem

Control

Linear is Optimal, Convex Problem

Radner
(Team Decision Problems)

Nested Information Pattern
[Witsenhausen, Ho, et al.]

Quadratic Invariance
[Swigart, Lall]

Spectral Factorization
[Lessard, Lall]

Partial Ordering Set
[Shah, Perrilo]

Implicit communication is not helpful

Communication

Nonlinear is Optimal, Nonconvex Problem

Witsenhausen
(Counterexample)

negative results or computer-based search …

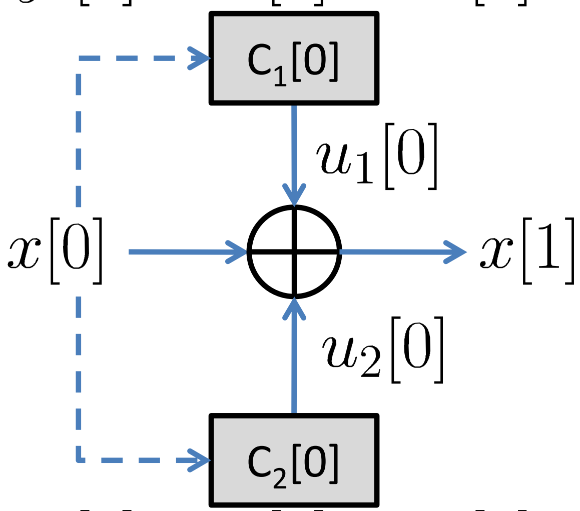
Approximated Optimal Solution
[Grover]

Implicit communication is helpful

History

- Radner's Problem

$$y_1[0] = x[0] + v_1[0]$$



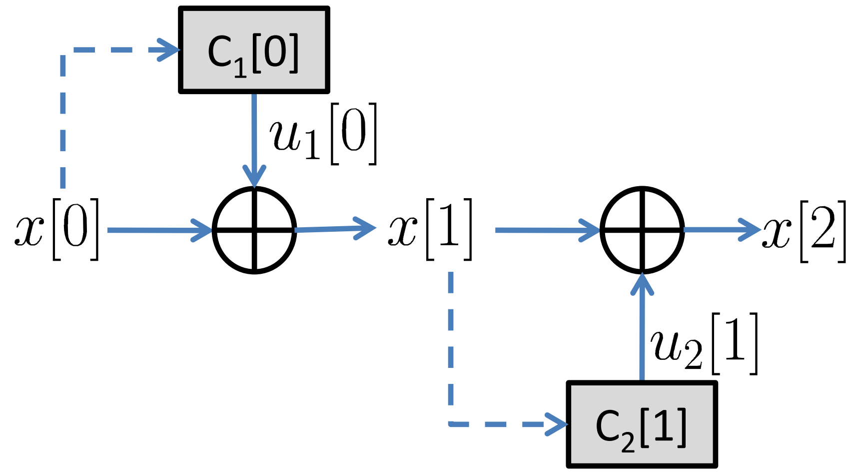
$$y_2[0] = x[0] + v_2[0]$$

$$\min_{u_1,u_2} \mathbb{E}[x[1]^2 + r_1 u_1^2[0] + r_2 u_2^2[0]]$$

- Linear is optimal

- Witsenhausen's Counterexample

$$y_1[0] = x[0] + v_1[0]$$



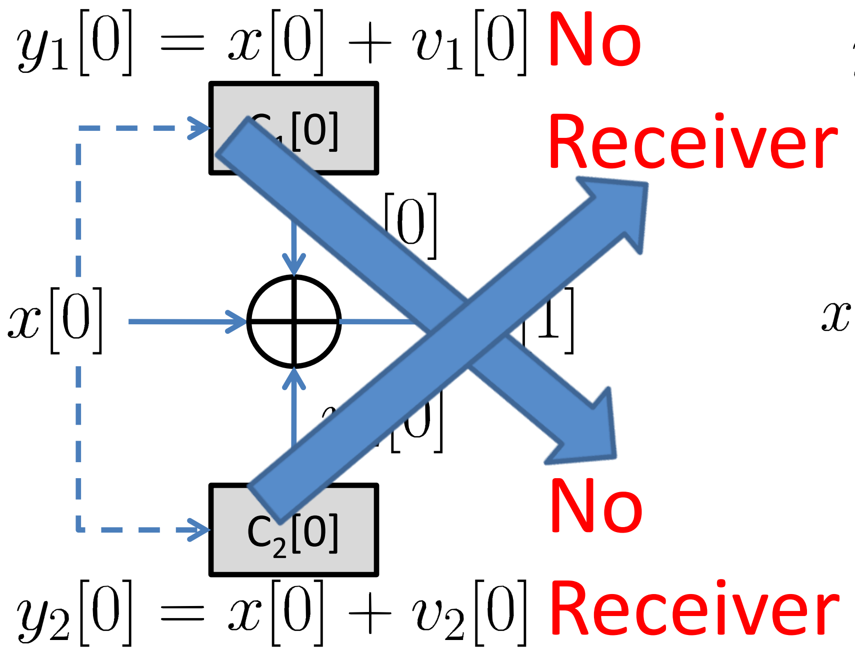
$$y_2[1] = x[1] + v_2[1]$$

$$\min_{u_1,u_2} \mathbb{E}[x[2]^2 + r_1 u_1^2[0] + r_2 u_2^2[1]]$$

- Linear is Not optimal

History

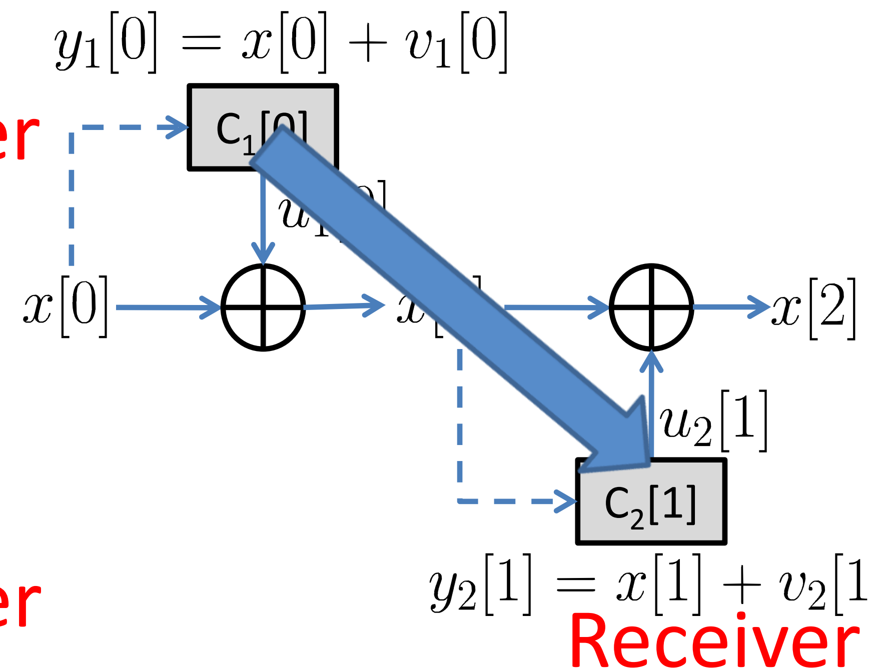
- Radner's Problem



$$\min_{u_1,u_2} \mathbb{E}[x[1]^2 + r_1 u_1^2[0] + r_2 u_2^2[0]]$$

- Linear is optimal
- Implicit Communication is Impossible

- Witsenhausen's Counterexample



$$\min_{u_1,u_2} \mathbb{E}[x[2]^2 + r_1 u_1^2[0] + r_2 u_2^2[1]]$$

- Linear is Not optimal
- Implicit Communication is possible

Simplest Infinite-horizon LQG Problem

General Infinite-horizon Decentralized LQG Problem

$$x[n+1] = Ax[n] + B_1 u_1[n] + \cdots + B_m u_m[n] + w[n]$$
$$y_1[n] = C_1 x[n] + v_1[n]$$
$$\vdots$$
$$y_m[n] = C_m x[n] + v_m[n]$$
$$\inf_{u_i} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[x^*[n]Qx[n] + \sum_{1\le i\le m} u_i^*[n]R_i u_i[n]]$$

Simplest Infinite-horizon Decentralized LQG Problem:

Two Controller, Scalar Plant

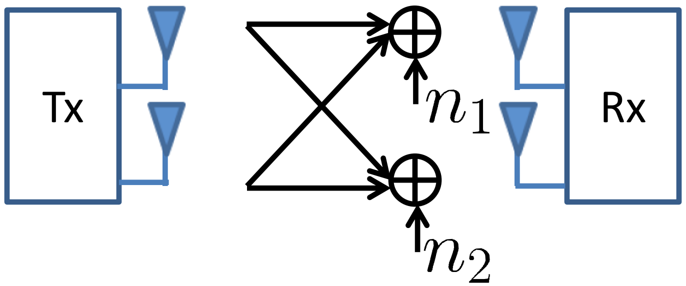
$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n] + v_1[n]$$
$$y_2[n] = x[n] + v_2[n]$$
$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n] + r_2 u_2^2[n]]$$

Learn from Wireless Communication Theory

Decentralized LQG Problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n] + v_1[n]$$
$$y_2[n] = x[n] + v_2[n]$$

MIMO Communication Problem



Linear Super-position of Signals, Gaussian Disturbance

Divide Cases:

(1) Fast Dynamics
- When $|a| \geq 4$

(2) Slow Dynamics
- When $|a| < 4$

Divide Cases:

(1) High-SNR (Signal-to-Noise Ratio)
- d.o.f. gain (rank of signal) is important
- Rank maximization scheme

(2) Low-SNR (Signal-to-Noise Ratio)
- Beam-forming gain (power of signal) is important
- Maximum-Ratio combining scheme

Fast Dynamics Case (When $|a| \geq 4$)

- Nonlinear Controller can infinitely outperform Linear Controller
- We will propose approximately optimal finite dimensional nonlinear controller.
- Main machinery: binary deterministic model

Binary Deterministic Model (Avestimehr et al.)

- Main idea: Write a number in binary expansion
  - Ex) Real Numbers

$$1/2 = 0.1$$
$$\pi = 11.001001\ldots$$

Binary Deterministic Model (Avestimehr et al.)

- Main idea: Write a number in binary expansion
  - Ex) Real Numbers

$$1/2 = 0.1$$
$$\pi = 11.001001\ldots$$

  - Ex) Random variables

$$Unif[0,1] = 0.b_1 b_2 b_3 \ldots$$
$$Unif[0,4] = b_1 b_2 . b_3 b_4 b_5 \ldots$$
$$\text{where } b_i \text{ are i.i.d. Bernoulli } 1/2$$

Binary Deterministic Model (Avestimehr et al.)

- Second Idea: Approximate Gaussian r.v. by Uniform r.v.

$$Unif[0,4] = b_1 b_2 . b_3 \ldots$$

$$\approx$$

$$\mathcal{N}(0,4^2) = b_1 b_2 . b_3 \ldots$$

$$\text{Gaussian with zero mean and variance } 4^2$$

Binary Deterministic Model (Avestimehr et al.)

- Third idea: Ignore Carry in Addition and Subtraction

Let $A \sim \mathcal{N}(0, 4^2)$ and $B \sim \mathcal{N}(0, 8^2)$

$$
\begin{array}{rl}
A = & a_1 a_2 . a_3 \ldots \\
+ \; B = & b_1 b_2 b_3 . b_4 \ldots \\
\hline
\end{array}
$$

$$A + B = b_1 (a_1 \oplus b_2)(a_2 \oplus b_3).(a_3 \oplus b_4)\ldots$$

Binary Deterministic Model (Avestimehr et al.)

- Third idea: Ignore Carry in Addition and Subtraction

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$$\begin{array}{r} A = \quad a_1 a_2 . a_3 \ldots \\ + \; B = b_1 b_2 b_3 . b_4 \ldots \\ \hline \end{array}$$

$$A + B = b_1 (a_1 \oplus b_2)(a_2 \oplus b_3).(a_3 \oplus b_4)\ldots$$

$$\begin{array}{r} A = \quad a_1 a_2 . a_3 \ldots \\ - \; B = b_1 b_2 b_3 . b_4 \ldots \\ \hline \end{array}$$

$$A - B = b_1 (a_1 \oplus b_2)(a_2 \oplus b_3).(a_3 \oplus b_4)\ldots$$

Binary Deterministic Model (Avestimehr et al.)

- Fourth idea: Multiplication and Division by a constant is bit-shift.

$$\text{Let } A \sim \mathcal{N}(0, 4^2)$$

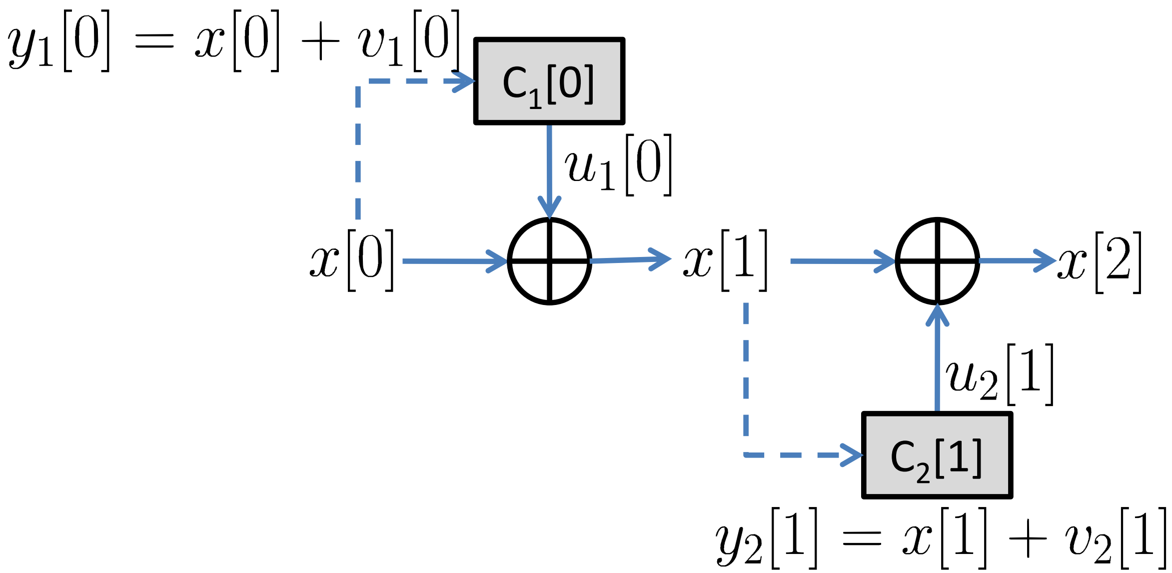
$$A = a_1 a_2 . a_3 \ldots$$

$$\text{Then}$$

$$4 \cdot A = a_1 a_2 a_3 a_4 . a_5 \ldots$$

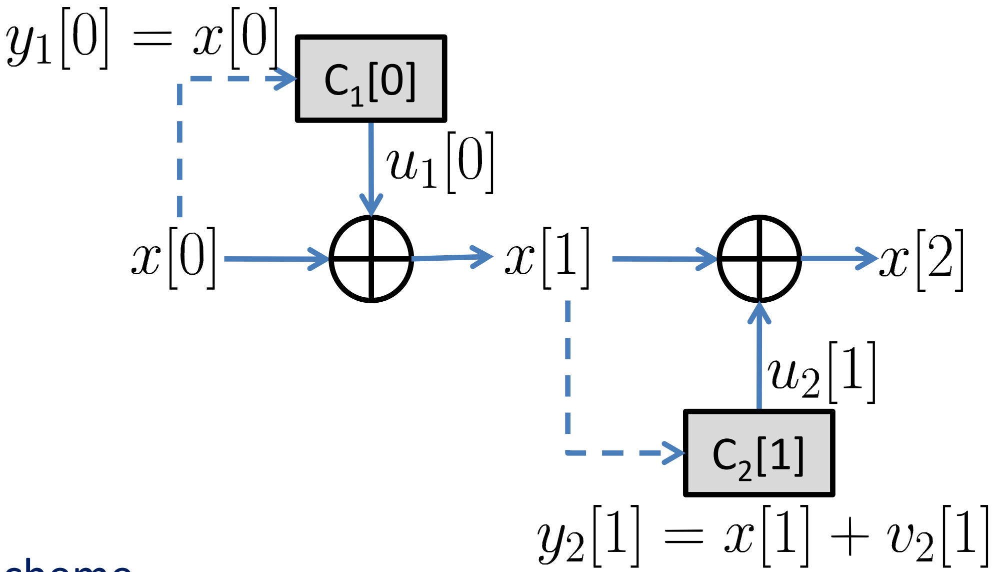
$$A/4 = 0.a_1 a_2 a_3 a_4 a_5 \ldots$$

Witsenhausen's Counterexample



$$\min_{u_1, u_2} \mathbb{E}[x[2]^2 + r_1 u_1^2[0] + r_2 u_2^2[1]]$$

Witsenhausen's Counterexample (Glover et al.)



$$x[0] \sim \mathcal{N}(0, 4^2)$$

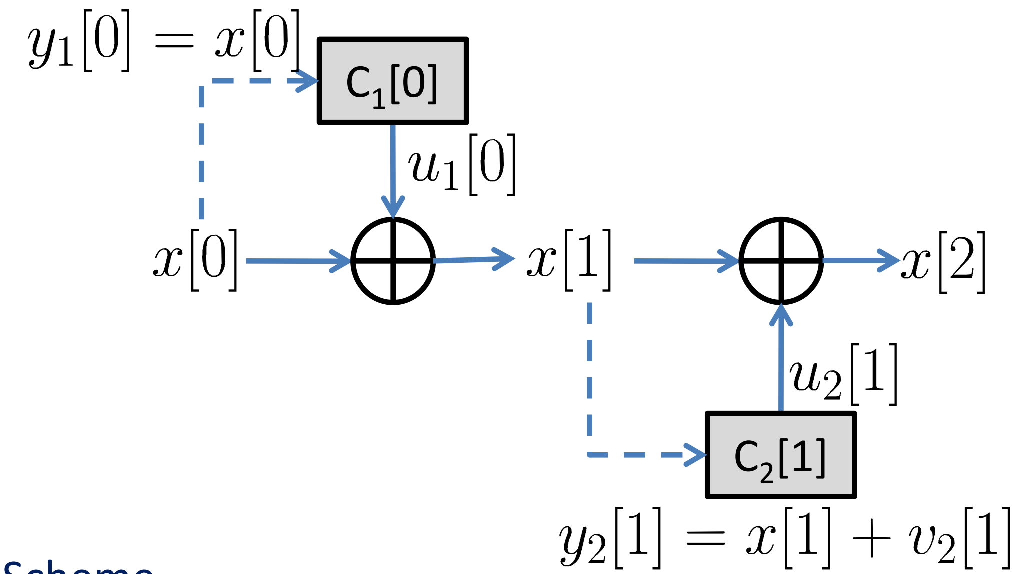
$$v_2[1] \sim \mathcal{N}(0, 1^2)$$

$$\min \mathbb{E}[x^2[2]]$$

$$\text{s.t. } \mathbb{E}[u_1^2[0]] \leq 1$$

Linear Scheme

Witsenhausen's Counterexample (Glover et al.)



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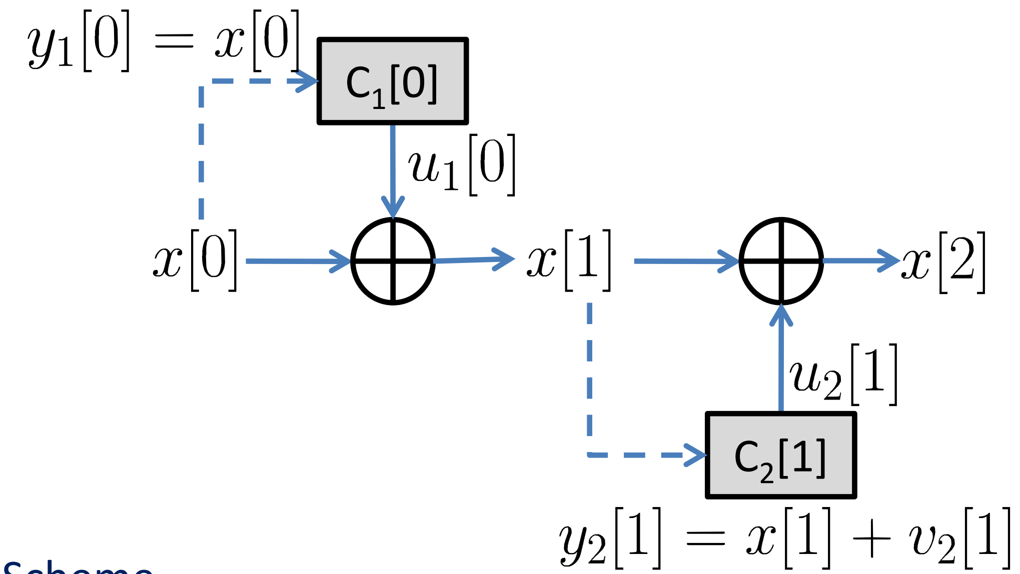
$$\text{s.t. } \mathbb{E}[u_1^2[0]] \leq 1$$

Linear Scheme

$x[0]$	$u_1[0]$
x_{01}	0
x_{02}	0
x_{03}	x_{01}
x_{04}	x_{02}

floating point

Witsenhausen's Counterexample (Glover et al.)



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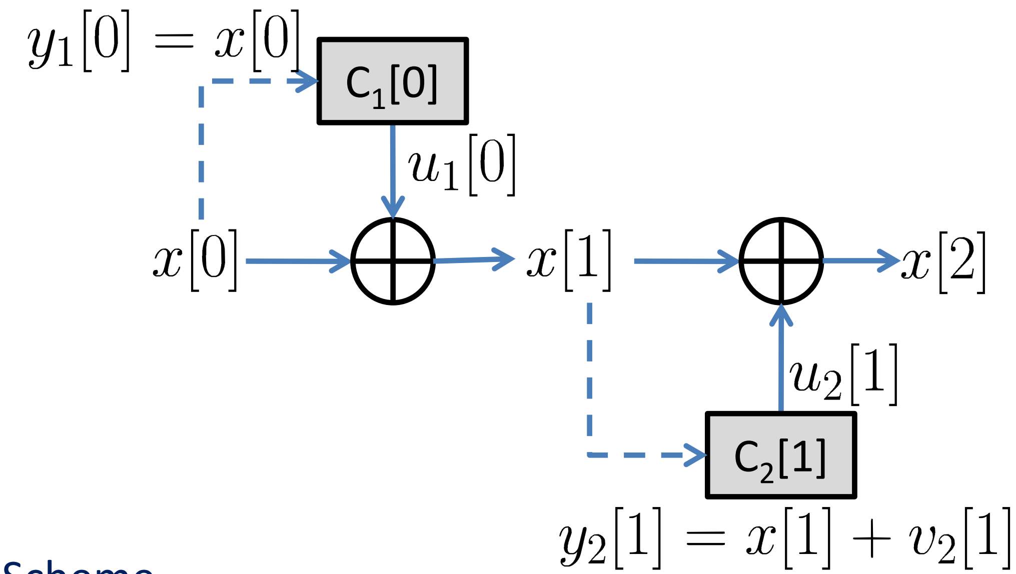
Linear Scheme

floating point

$x[0]$	$u_1[0]$	$x[1]$
x_{01}	0	x_{11}
x_{02}	0	x_{12}
x_{03}	x_{01}	x_{13}
x_{04}	x_{02}	x_{14}

$$y_2[1] = x_{11}x_{12}.(x_{13} \oplus v_{21})(x_{13} \oplus v_{22})$$

Witsenhausen's Counterexample (Glover et al.)



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Linear Scheme

	$x[0]$	$u_1[0]$	$x[1]$	$u_2[1]$	$x[2]$
	x_{01}	0	x_{11}	x_{11}	0
	x_{02}	0	x_{12}	x_{12}	0
floating point	x_{03}	x_{01}	x_{13}	$x_{13} \oplus v_{21}$	v_{21}
	x_{04}	x_{02}	x_{14}	$x_{14} \oplus v_{22}$	v_{22}

$$y_2[1] = x_{11}x_{12}.(x_{13} \oplus v_{21})(x_{13} \oplus v_{22})$$

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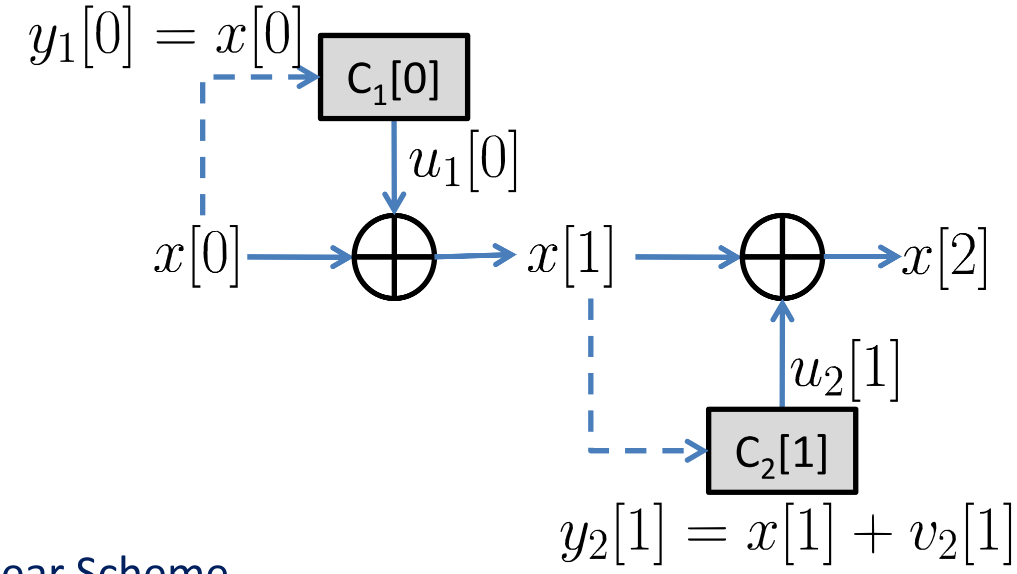

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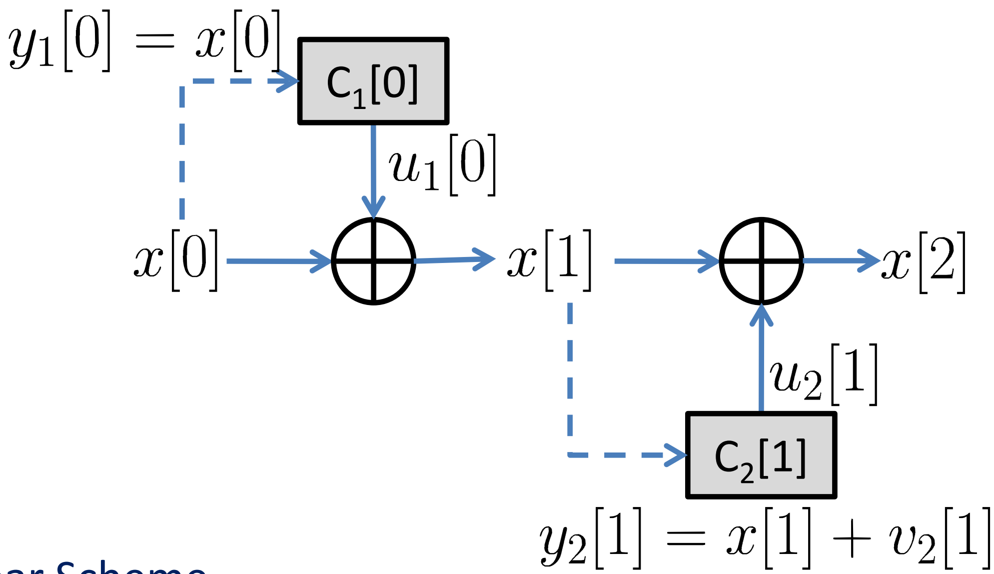
Nonlinear Scheme

	$x[0]$	$u_1[0]$
	x_{01}	0
	x_{02}	0
floating point		
	x_{03}	x_{03}
	x_{04}	x_{04}

Witsenhausen's Counterexample (Glover et al.)



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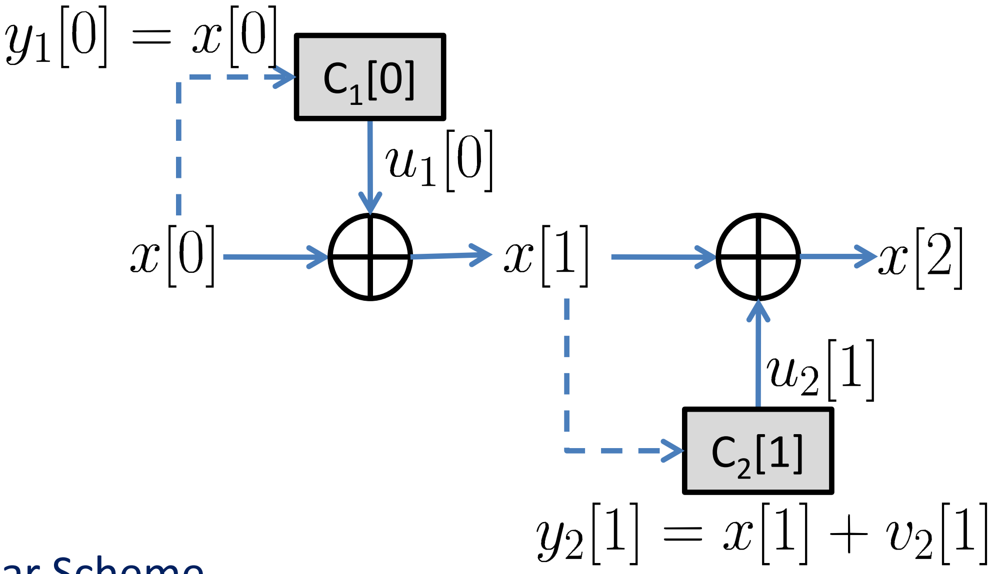
Nonlinear Scheme

$x[0]$	$u_1[0]$	$x[1]$
x_{01}	0	x_{11}
x_{02}	0	x_{12}
x_{03}	x_{03}	0
x_{04}	x_{04}	0

floating point

$$y_2[1] = x_{11}x_{12}.v_{21}v_{22}$$

Witsenhausen's Counterexample (Glover et al.)



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Nonlinear Scheme

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	x_{02}	0	x_{12}	x_{12}	0
floating point					
	x_{03}	x_{03}	0	0	0
	x_{04}	x_{04}	0	0	0

$$y_2[1] = x_{11}x_{12}.v_{21}v_{22}$$

Witsenhausen's Counterexample (Glover et al.)

- From deterministic model to Reals

Controller 1

$$y_1[0] = x_{01}x_{02}.x_{03}x_{04} \Longrightarrow u_1[0] = 00.x_{03}x_{04}$$

Controller 2

$$y_2[1] = x_{11}x_{12}.v_{21}v_{22} \Longrightarrow u_2[1] = x_{11}x_{12}.00$$

Witsenhausen's Counterexample (Glover et al.)

- From deterministic model to Reals

Controller 1

$$y_1[0] = x_{01}x_{02}.x_{03}x_{04} \Rightarrow u_1[0] = 00.x_{03}x_{04}$$

$u_1[0]$ is the remainder of $y_1[0]$ divided by 1

$$u_1[0] := R_1(y_1[0])$$

Controller 2

$$y_2[1] = x_{11}x_{12}.v_{21}v_{22} \Rightarrow u_2[1] = x_{11}x_{12}.00$$

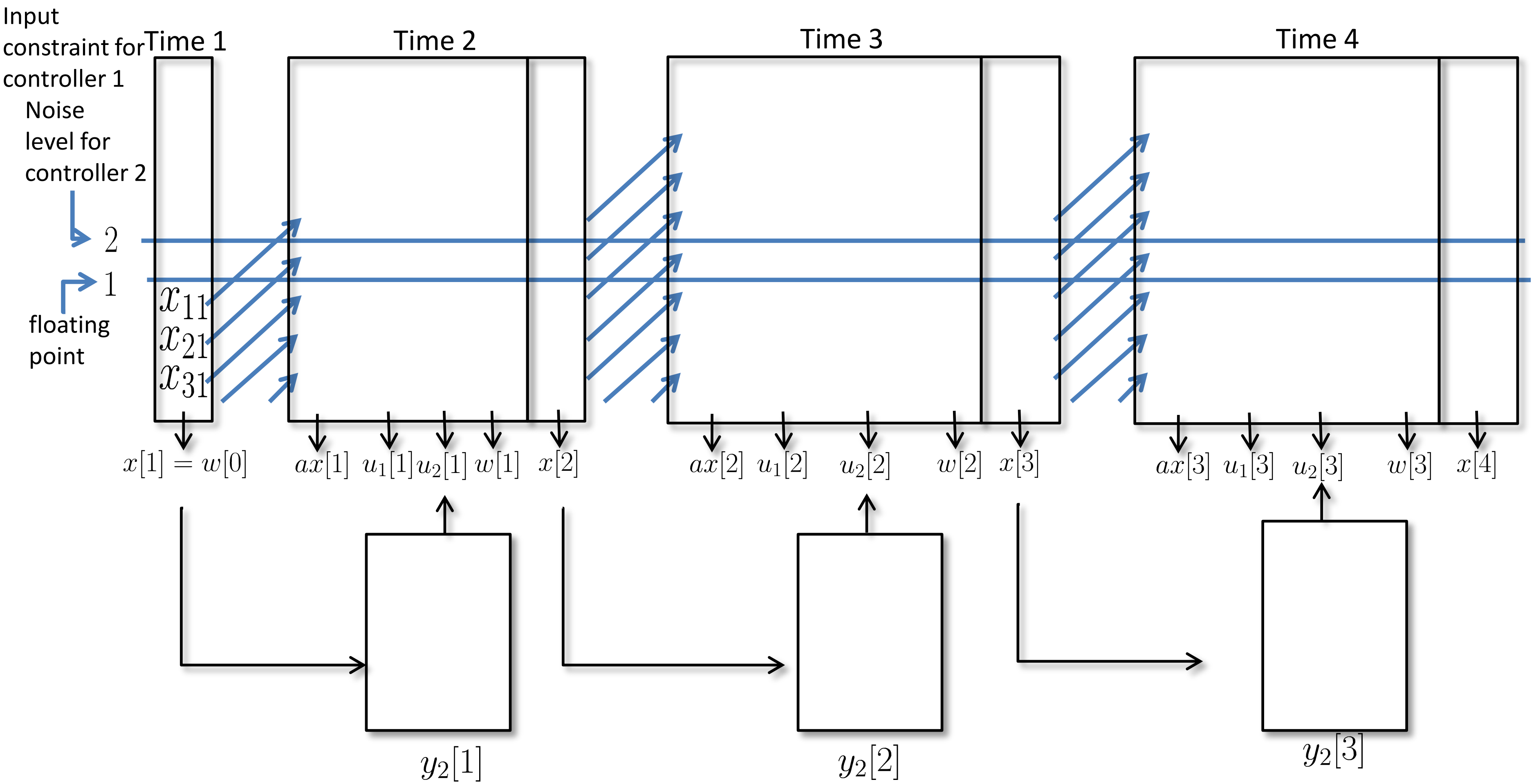
$u_2[1]$ is the quotient of $y_2[1]$ divided by 1

$$u_2[1] := Q_1(y_2[1])$$

Linear Controller

$$x[n+1] = 4x[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n]$$
$$y_2[n] = x[n] + v_2[n]$$

$$w[n] \sim \mathcal{N}(0,1)$$
$$v_2[n] \sim \mathcal{N}(0,2^2)$$
$$\mathbb{E}[u_1^2[n]] \leq 2^2$$



Linear Controller

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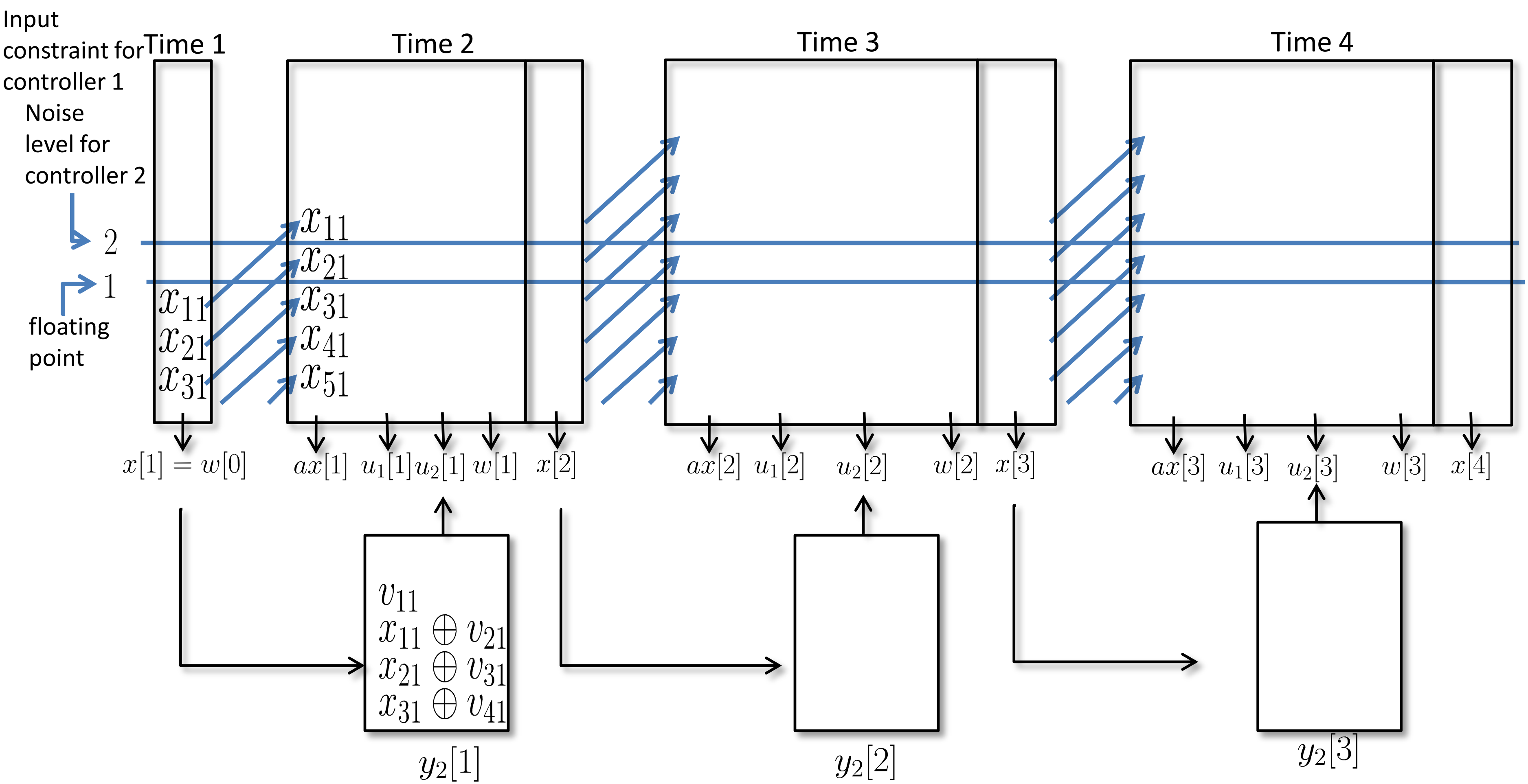

Linear Controller

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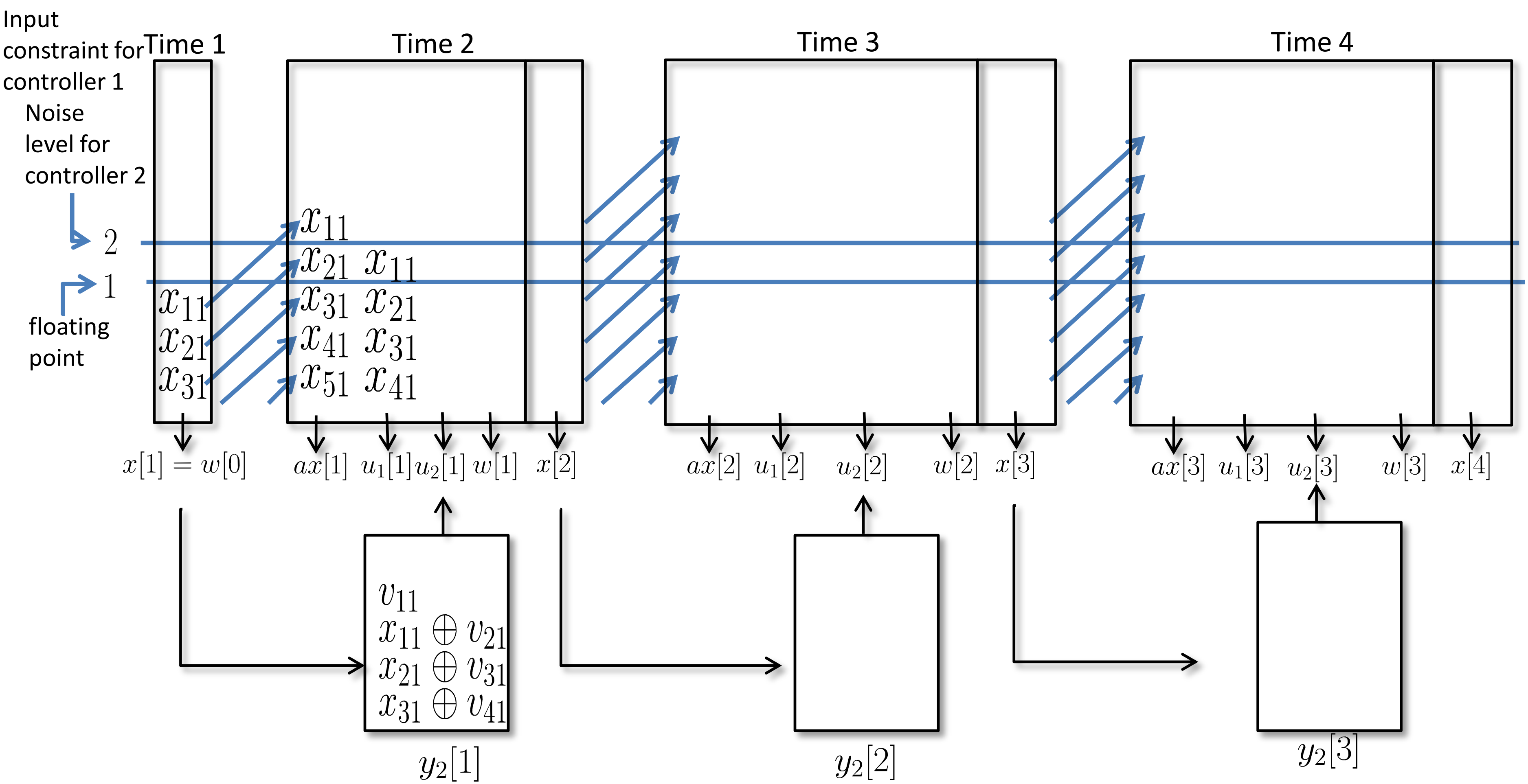
$$\mathbb{E}[u_1^2[n]] \leq 2^2$$



Linear Controller

$$x[n+1] = 4x[n] + u_1[n] + u_2[n] + w[n]$$
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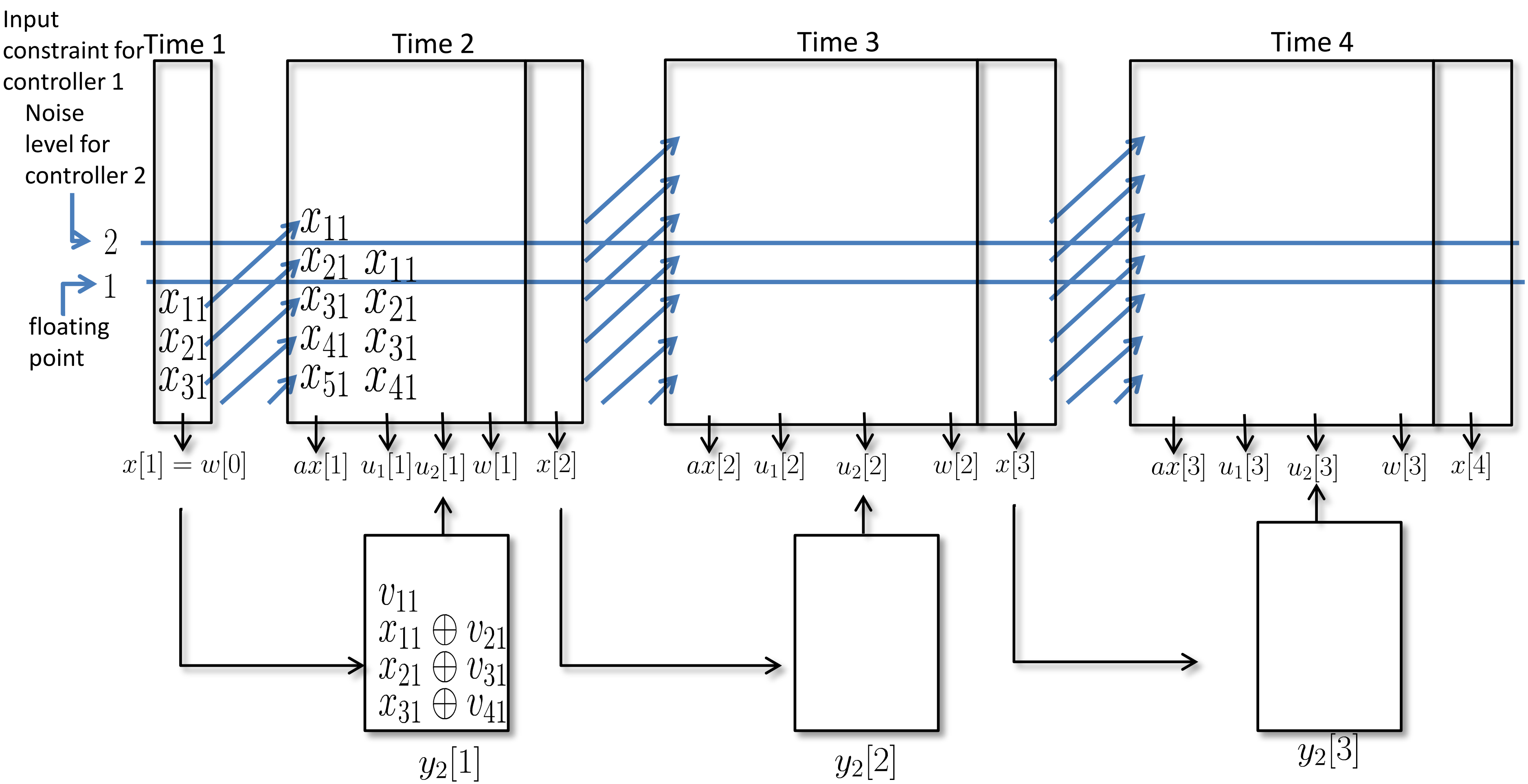

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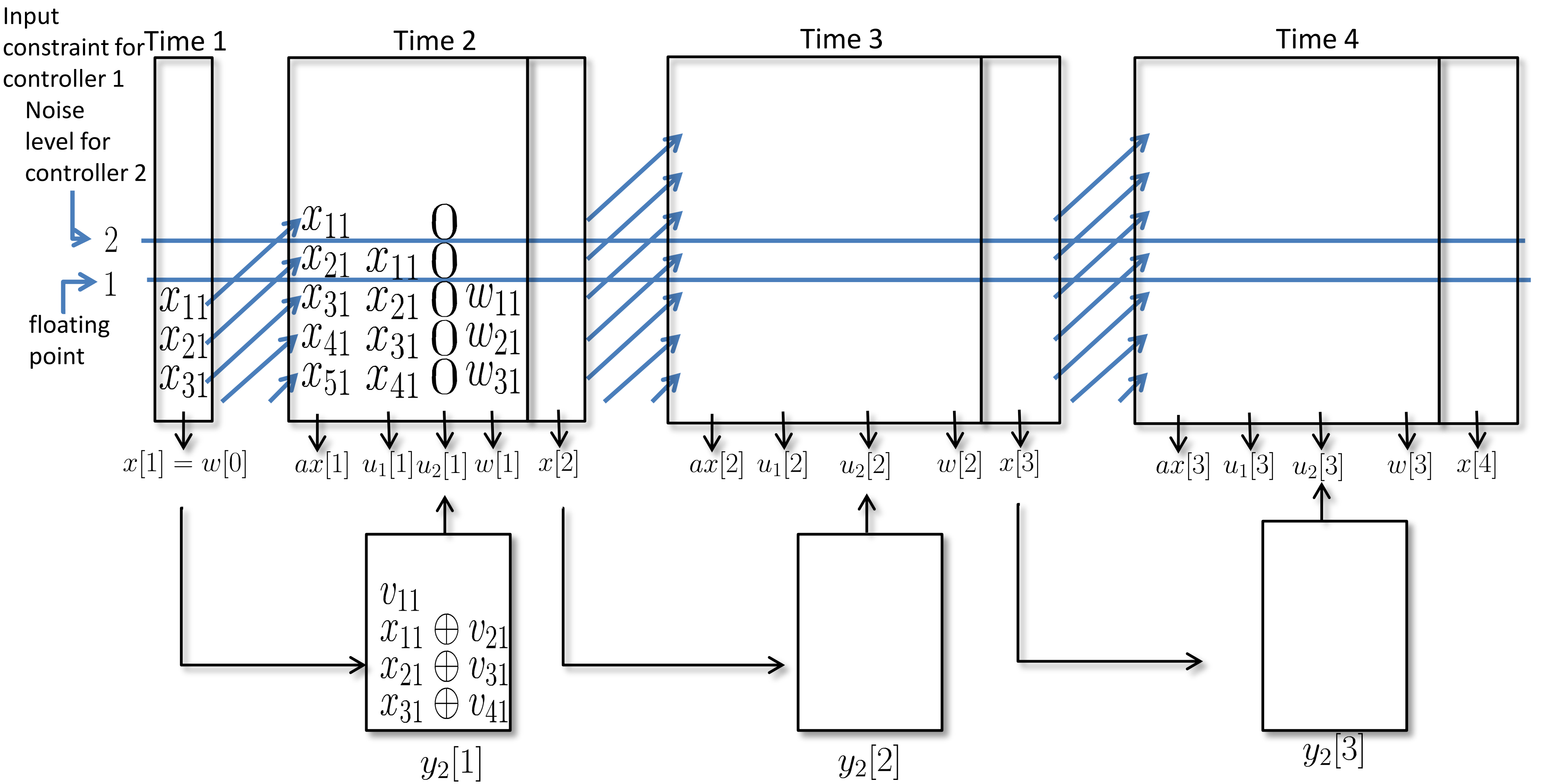

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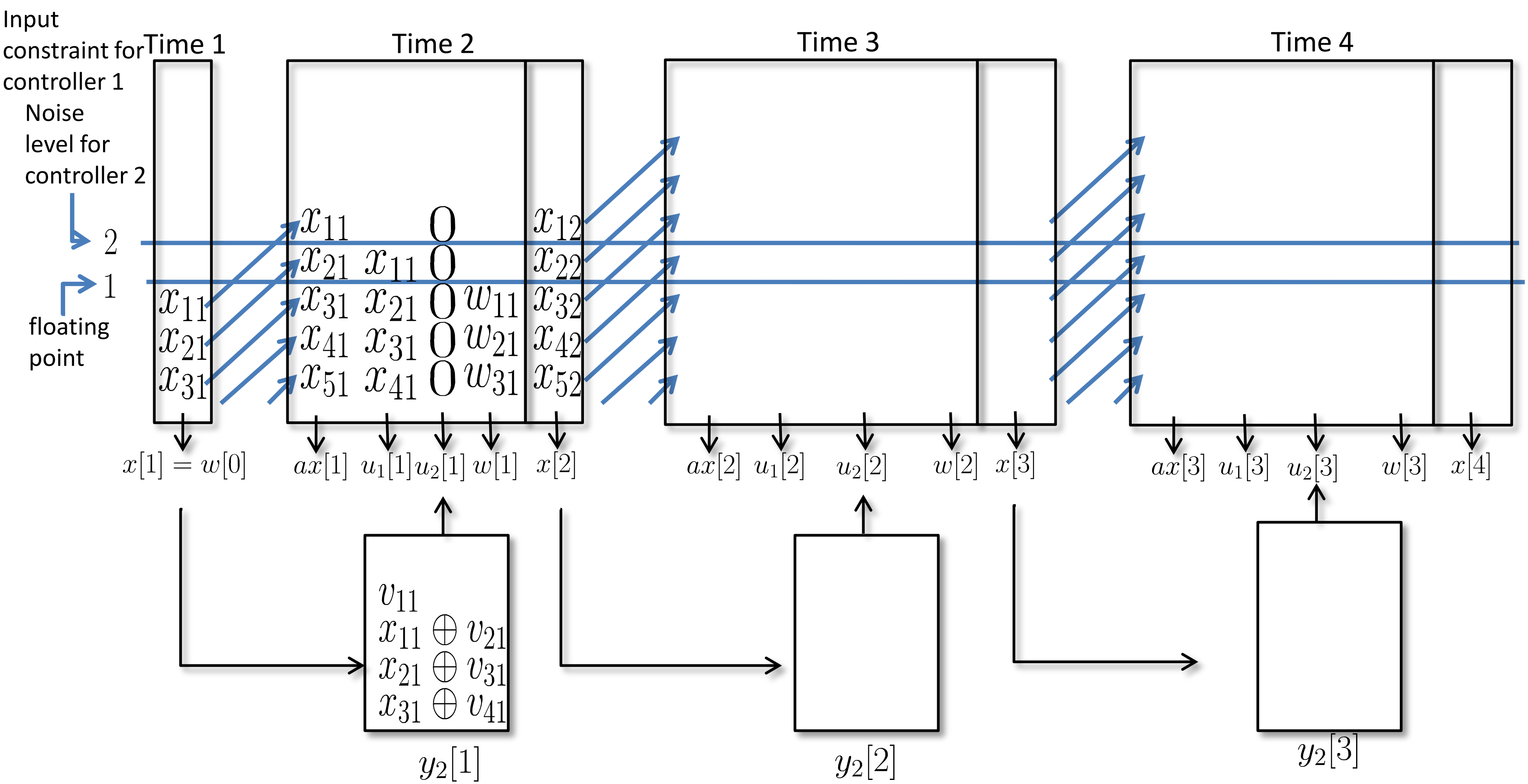

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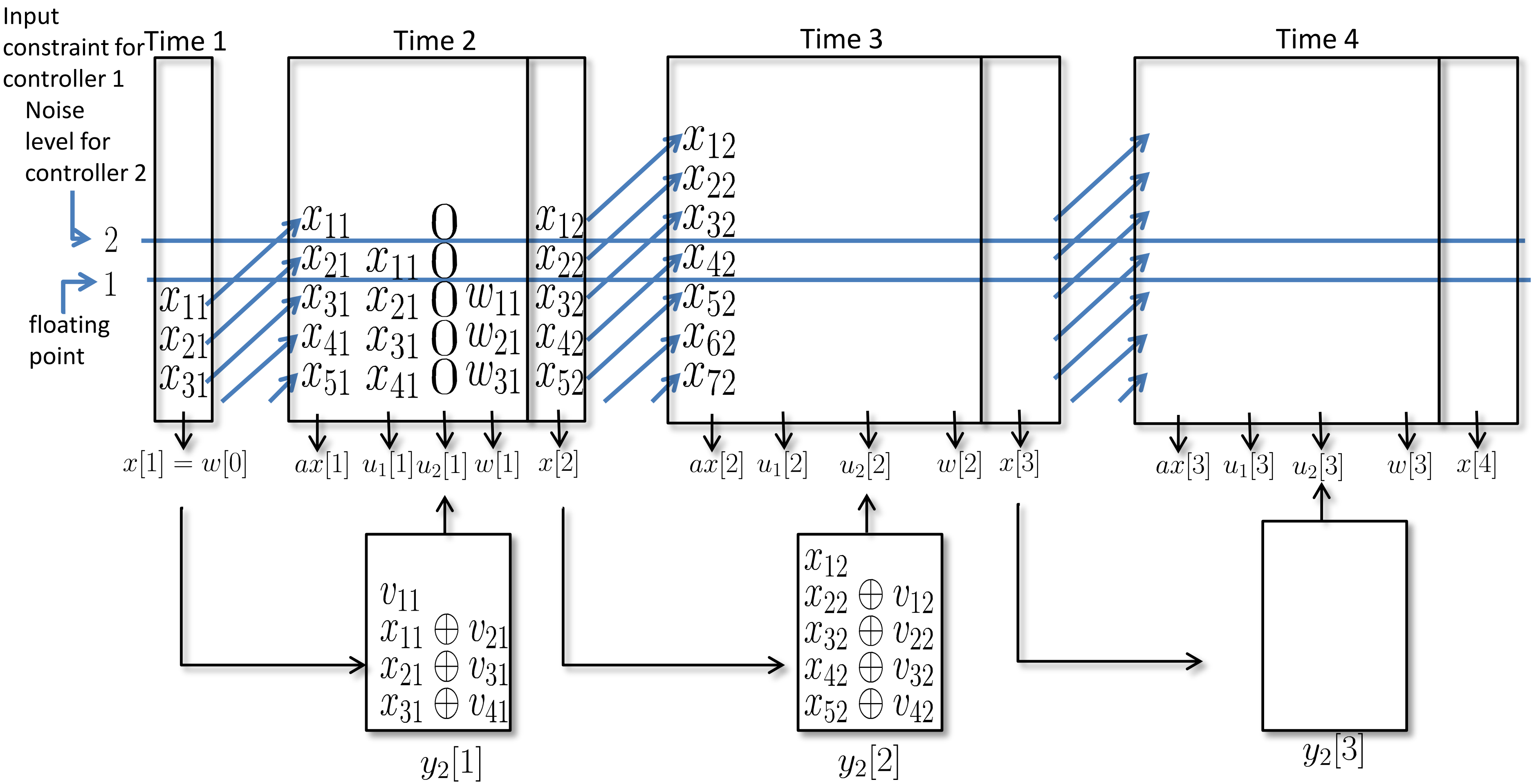

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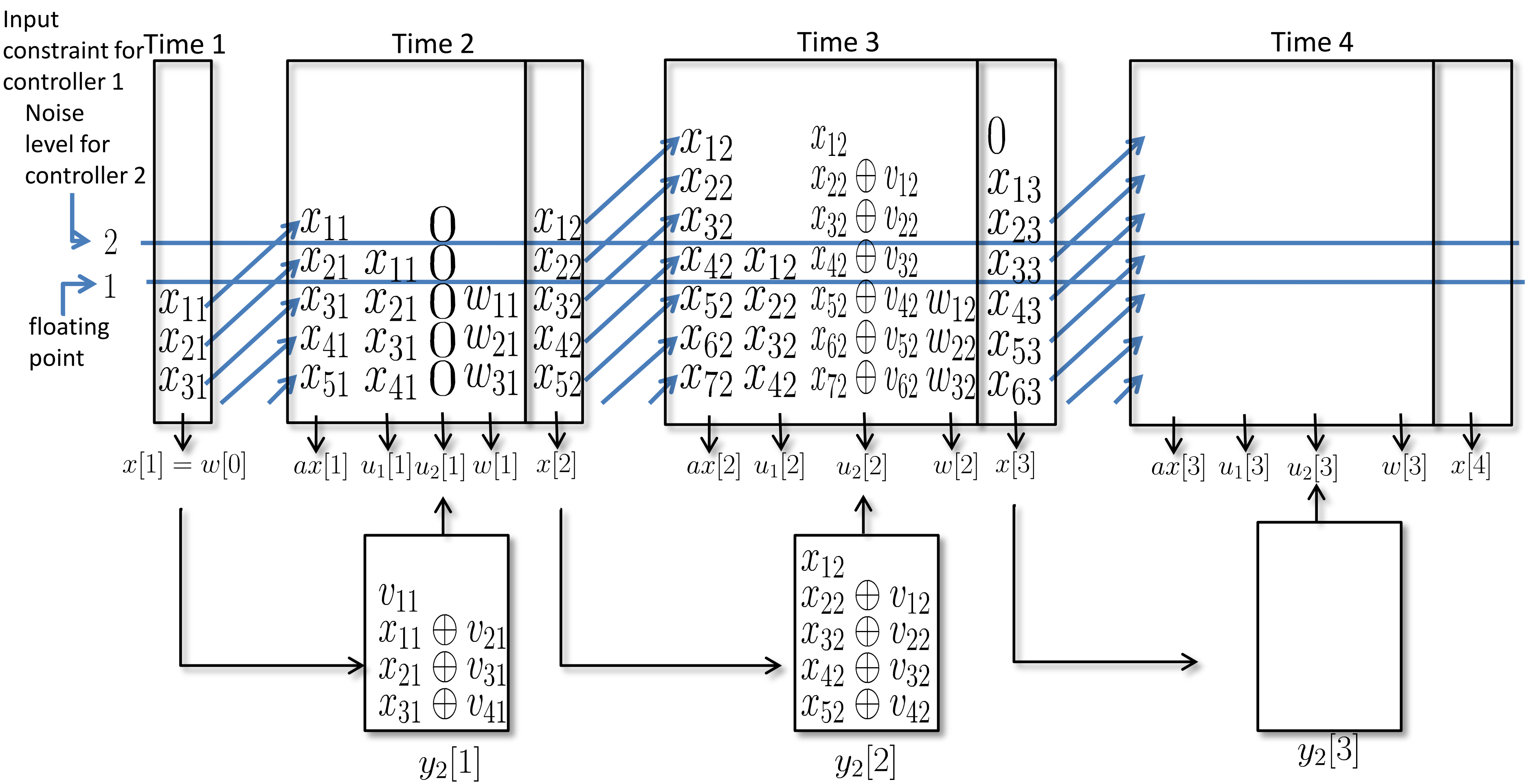

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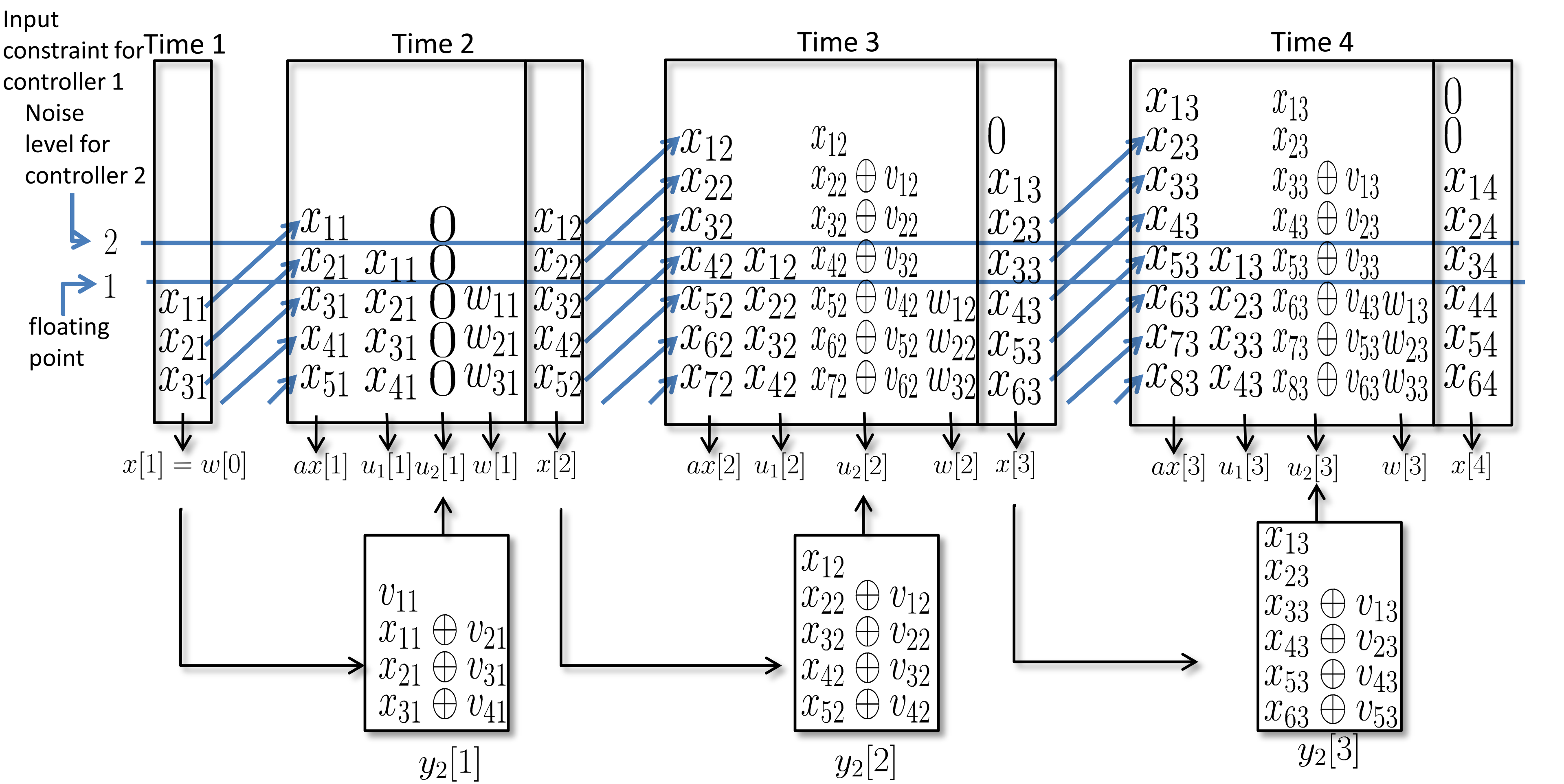

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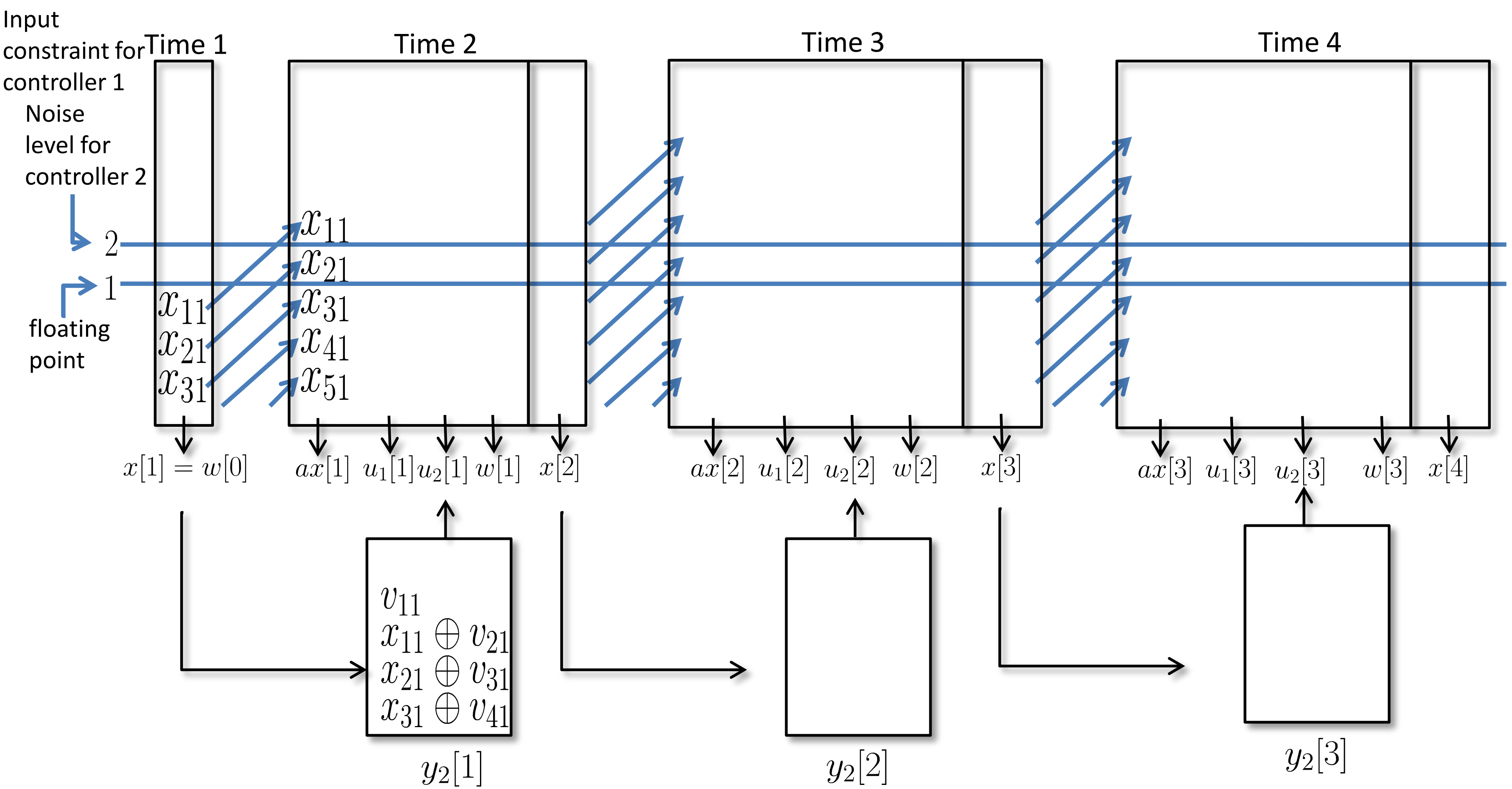

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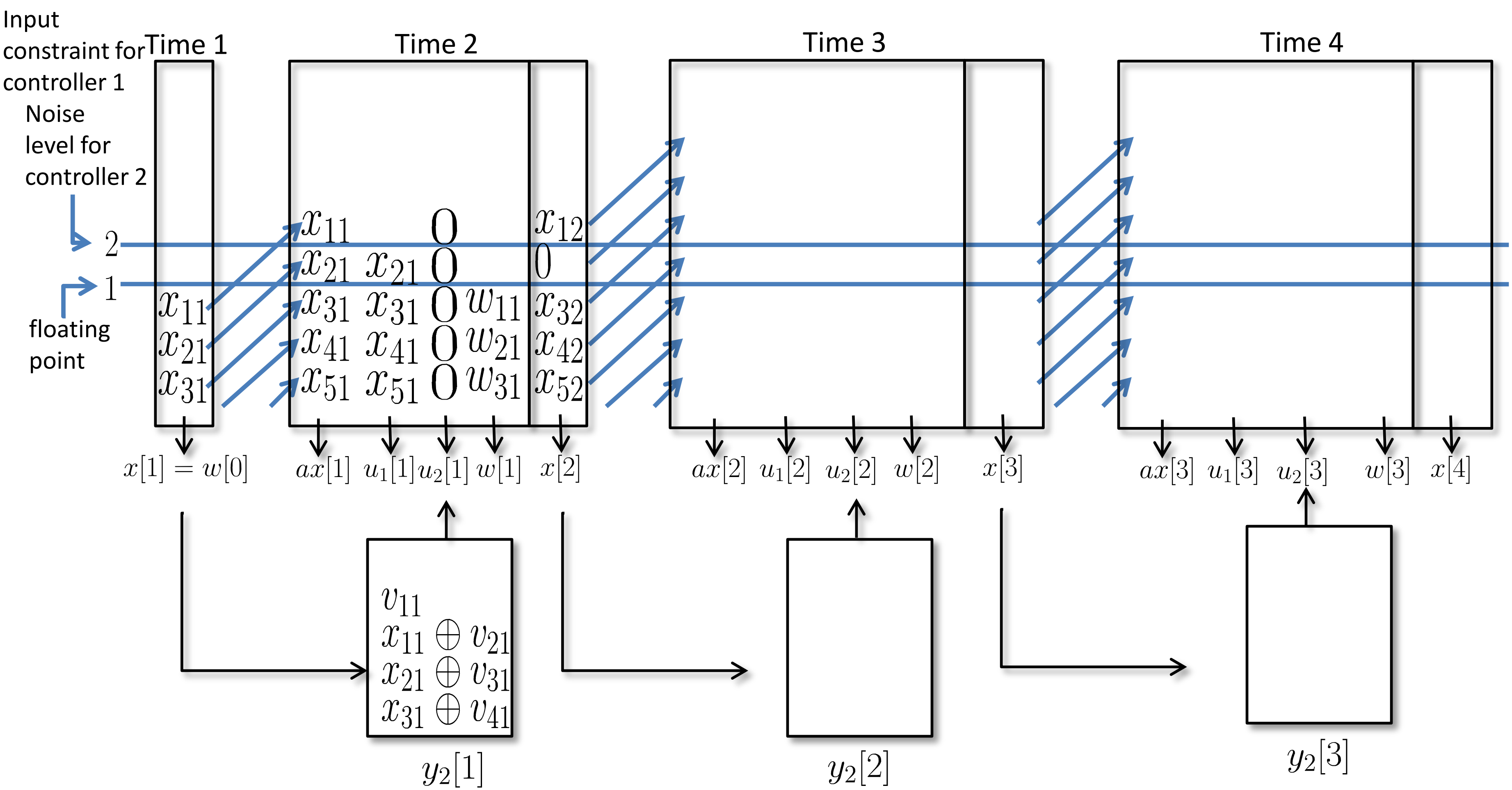

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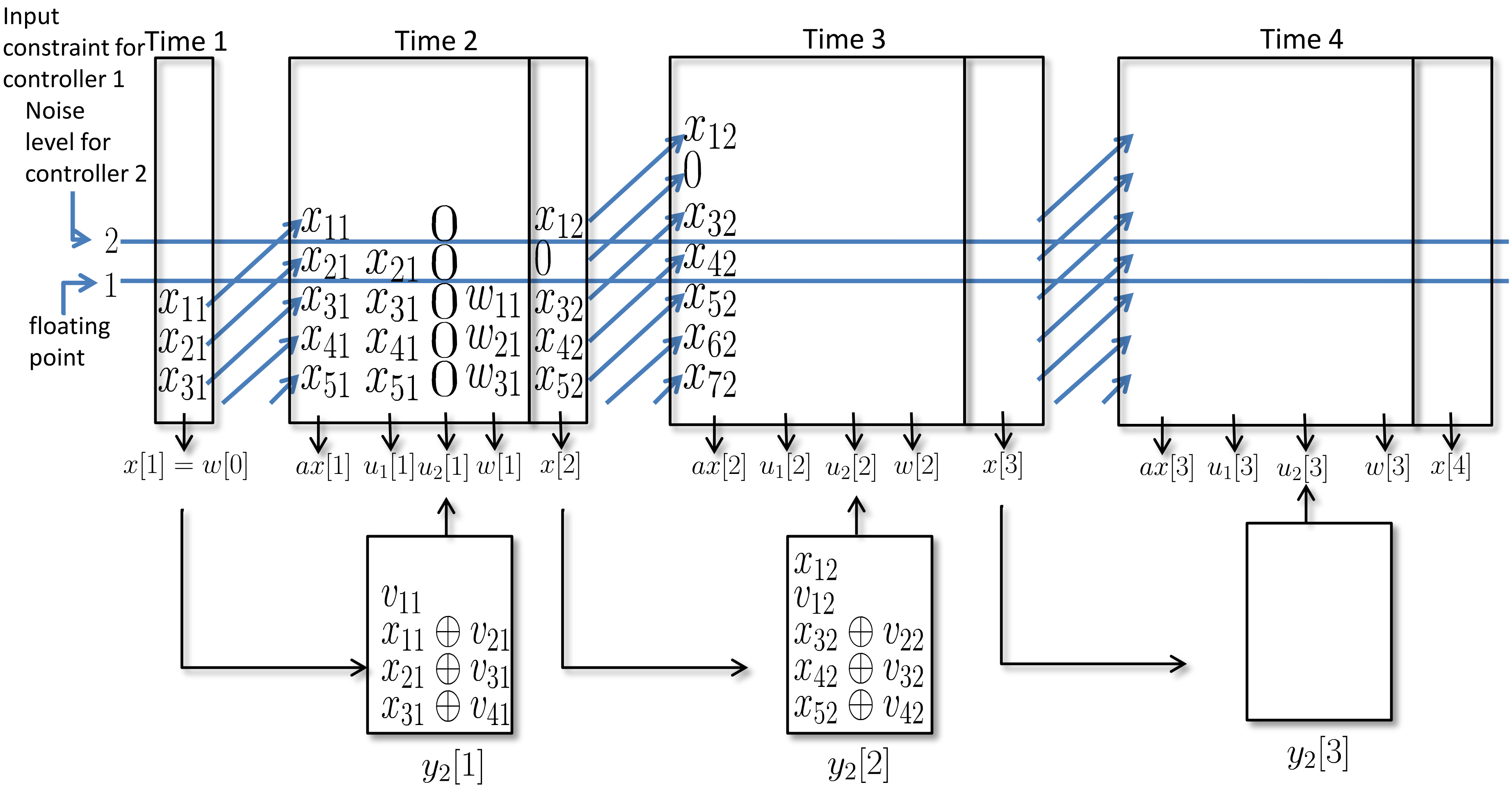

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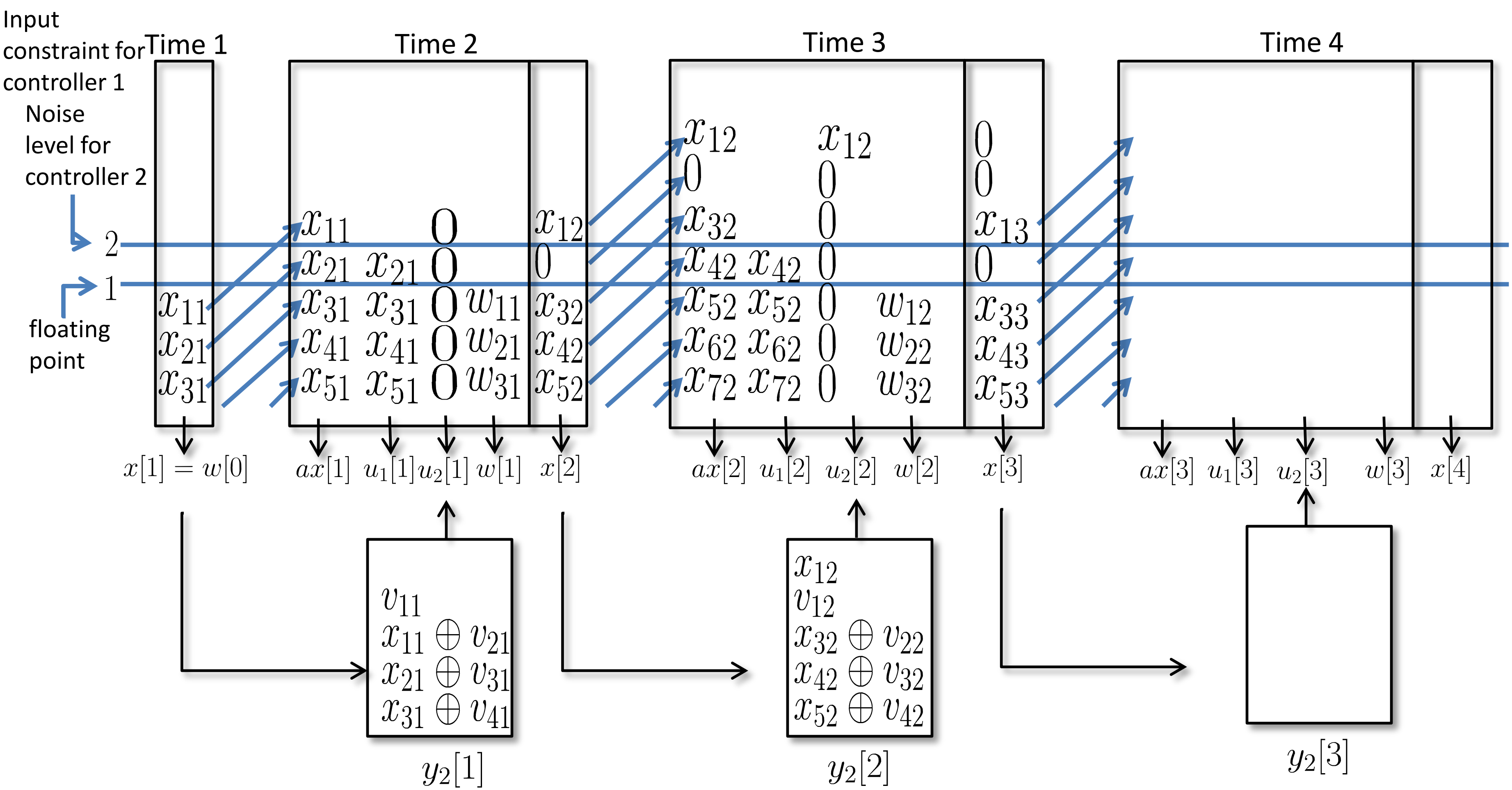

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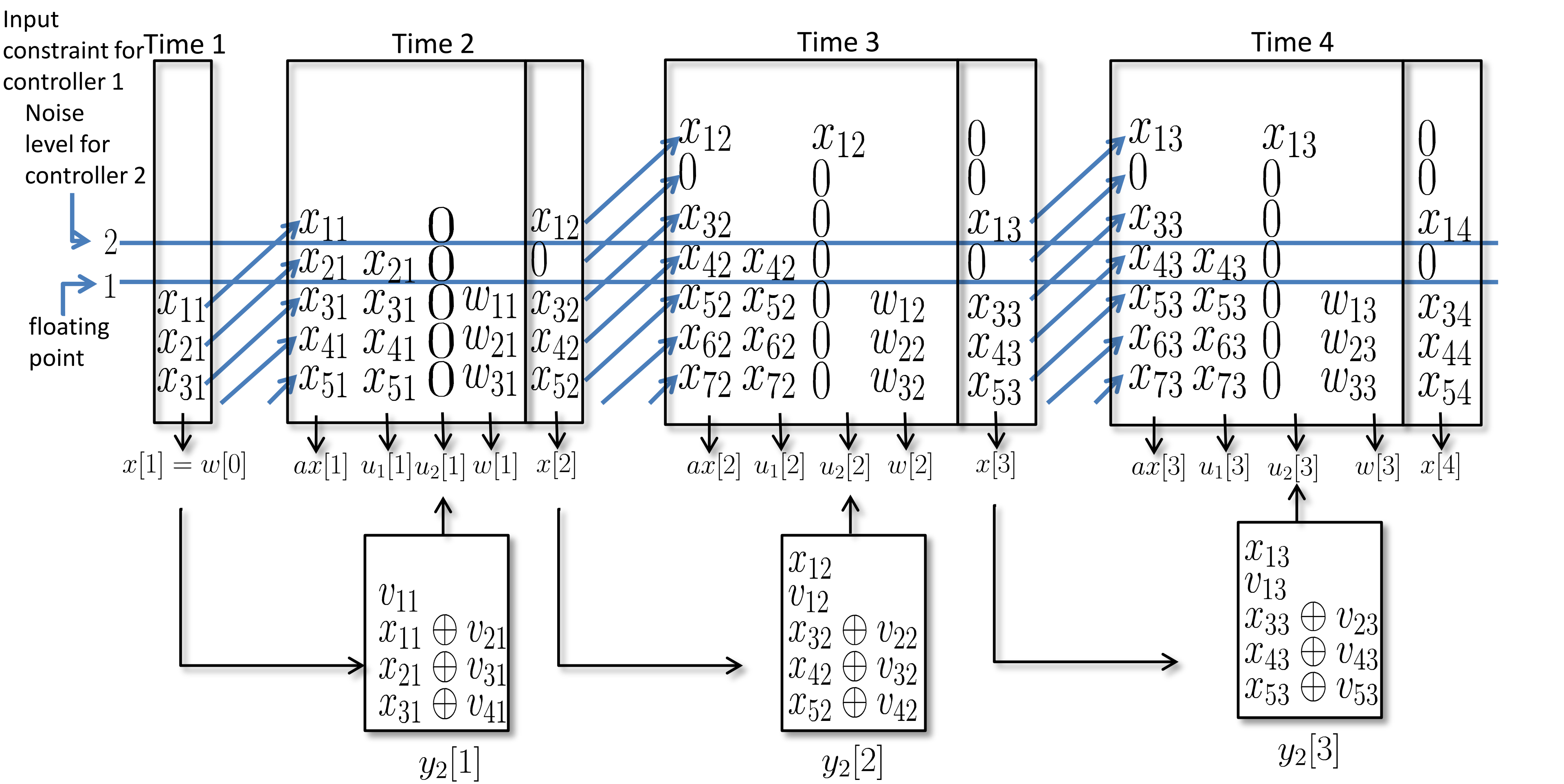
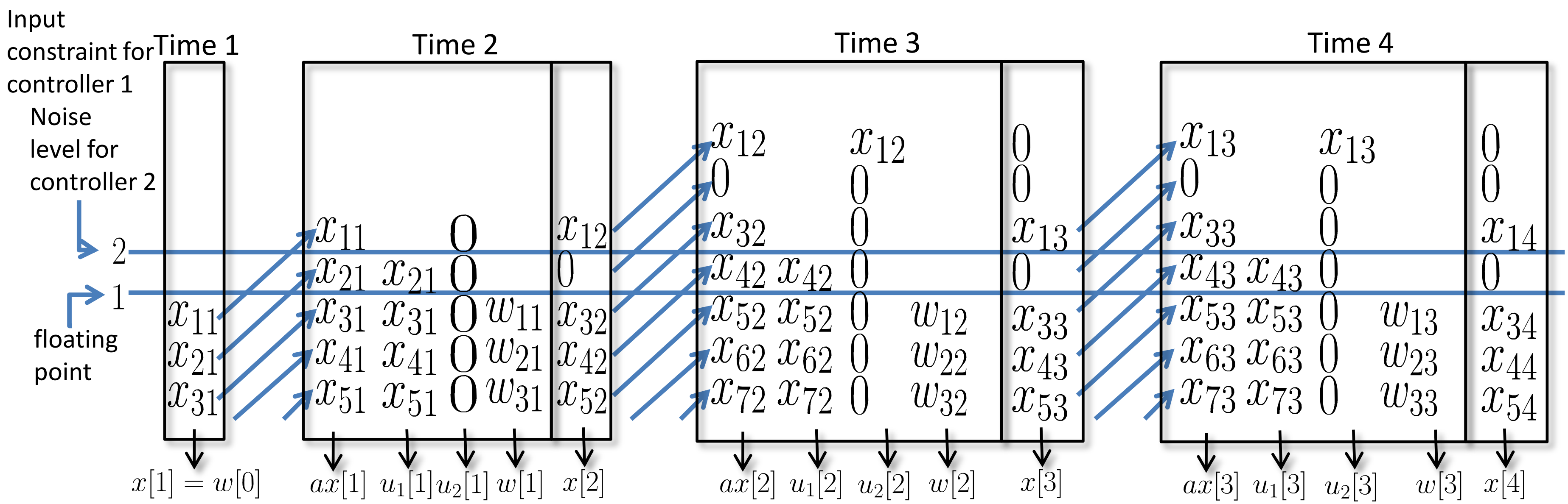


Nonlinear Controller

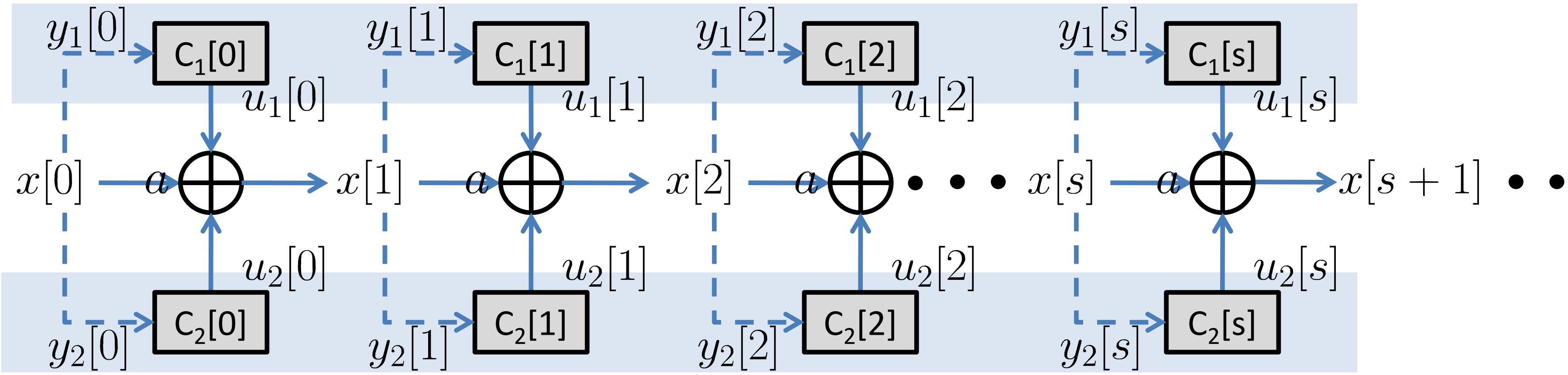


The corresponding scheme for the Reals

1-Stage Signaling Strategy $L_{sig,1}$

$$u_1[n] = -aR_d(y_1[n])$$
$$u_2[n] = -a(Q_{ad}(y_2[n] - R_{ad}(u_2[n-1])) + R_{ad}(u_2[n-1]))$$

Approximately Optimal Strategy



$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$

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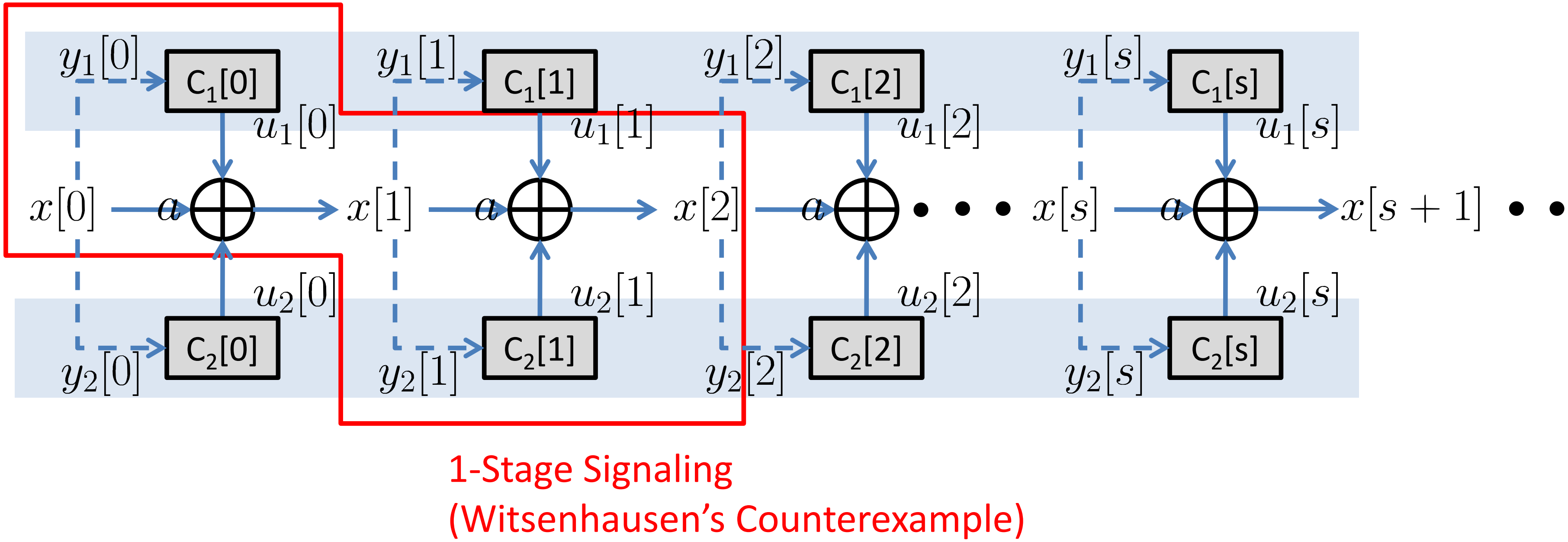
Approximately Optimal Strategy



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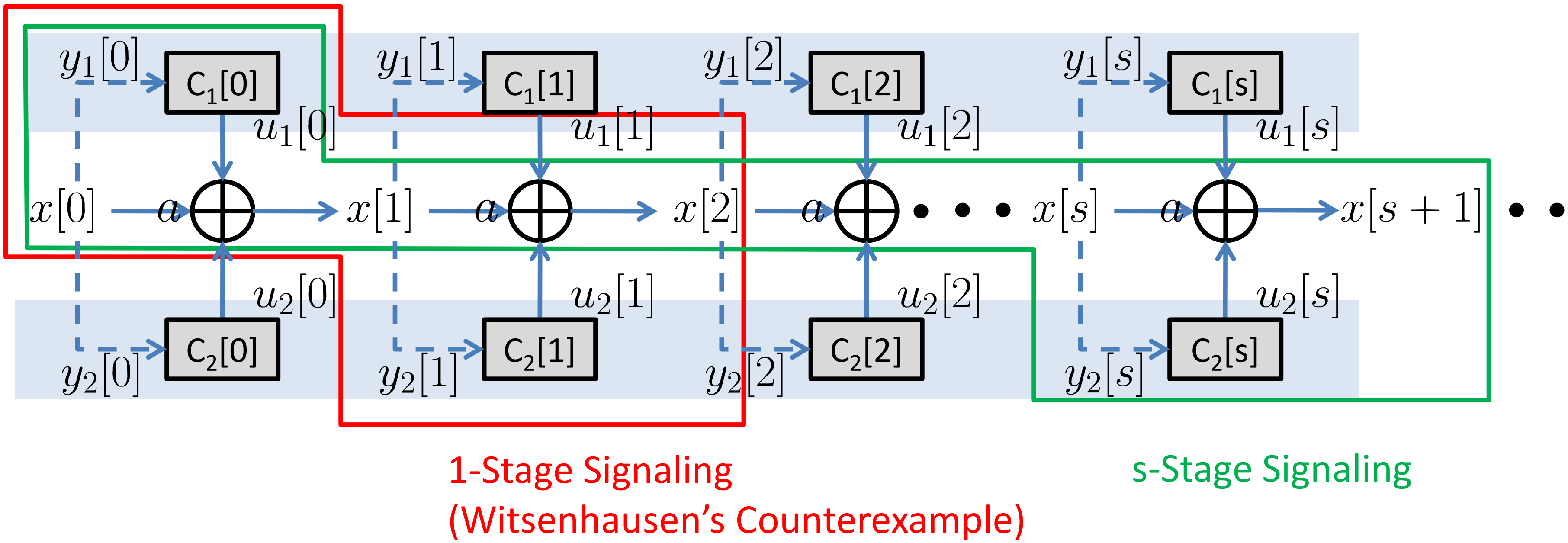


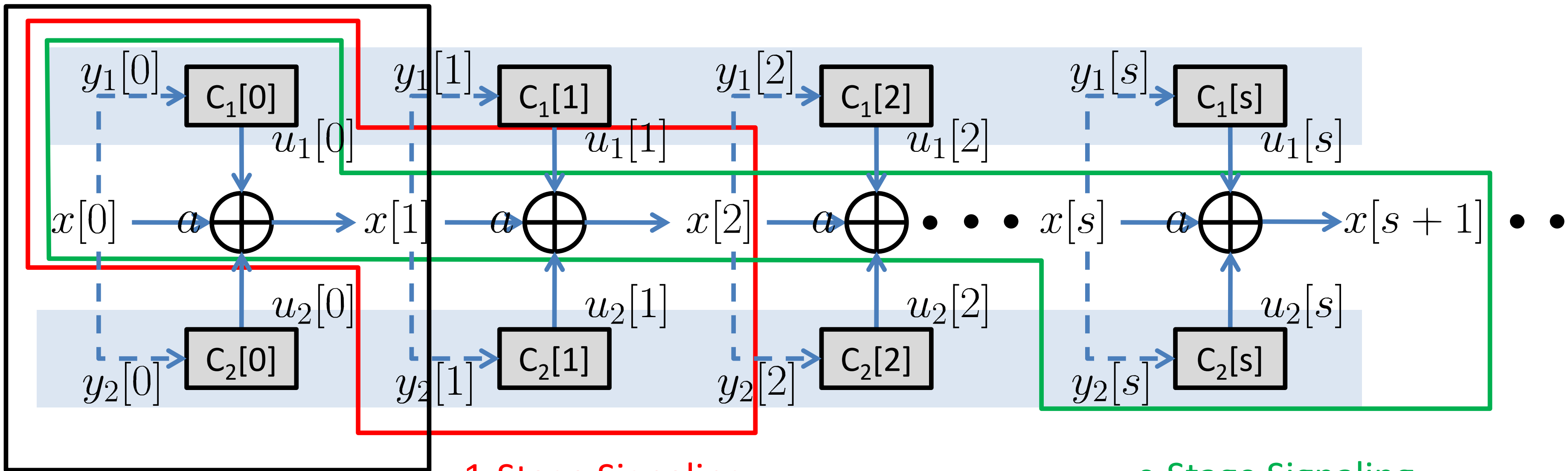
s-Stage Signaling Strategy $L_{sig,s}$

$$u_1[n] = -aR_d(y_1[n])$$

$$u_2[n] = -a(Q_{a^s d}(y_2[n] - R_{a^s d}(\sum_{1 \le i \le s} a^{i-1} u_2[n-i]))$$

$$+ R_{a^s d}(\sum_{1 \le i \le s} a^{i-1} u_2[n-i]))$$

Approximately Optimal Strategy



1-Stage Signaling
(Witsenhausen's Counterexample)

s-Stage Signaling

Radner's Problem
(0-Stage Signaling is impossible)

Linear Strategy L_{lin}

$$u_1[n] = 0$$
$$u_2[n] = -ay_2[n]$$

or

$$u_1[n] = -ay_1[n]$$
$$u_2[n] = 0$$

Approximately Optimal Strategy

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n] + v_1[n]$$
$$y_2[n] = x[n] + v_2[n]$$

Theorem [Park and Sahai, 2012]

There exists $c < 6 \cdot 10^6$ such that for all $|a| \geq 4$, q, r_1, r_2, σ_w, σ_{v1}, σ_{v2}

$$\frac{\inf_{u_1,u_2 \in L_{lin} \bigcup \cup_s L_{sig,s}} \limsup_{N\to\infty} \frac{1}{N} \sum_{0 \leq n < N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n] + r_2 u_2^2[n]]}{\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0 \leq n < N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n] + r_2 u_2^2[n]]} \leq c$$

Slow Dynamics Case (When $|a|<4$)

- SNR(Signal-to-Noise Ratio) for implicit communication between two controller is bounded by $|a|$.

- Single Controller Linear Strategy is optimal within a constant ratio.

- Unlike Fast Dynamics Case, we need Kalman filtering estimator.

Single Controller Optimal Strategy

$$x[n+1] = ax[n] + u[n] + w[n]$$
$$y[n] = x[n] + v[n]$$

Average Cost:

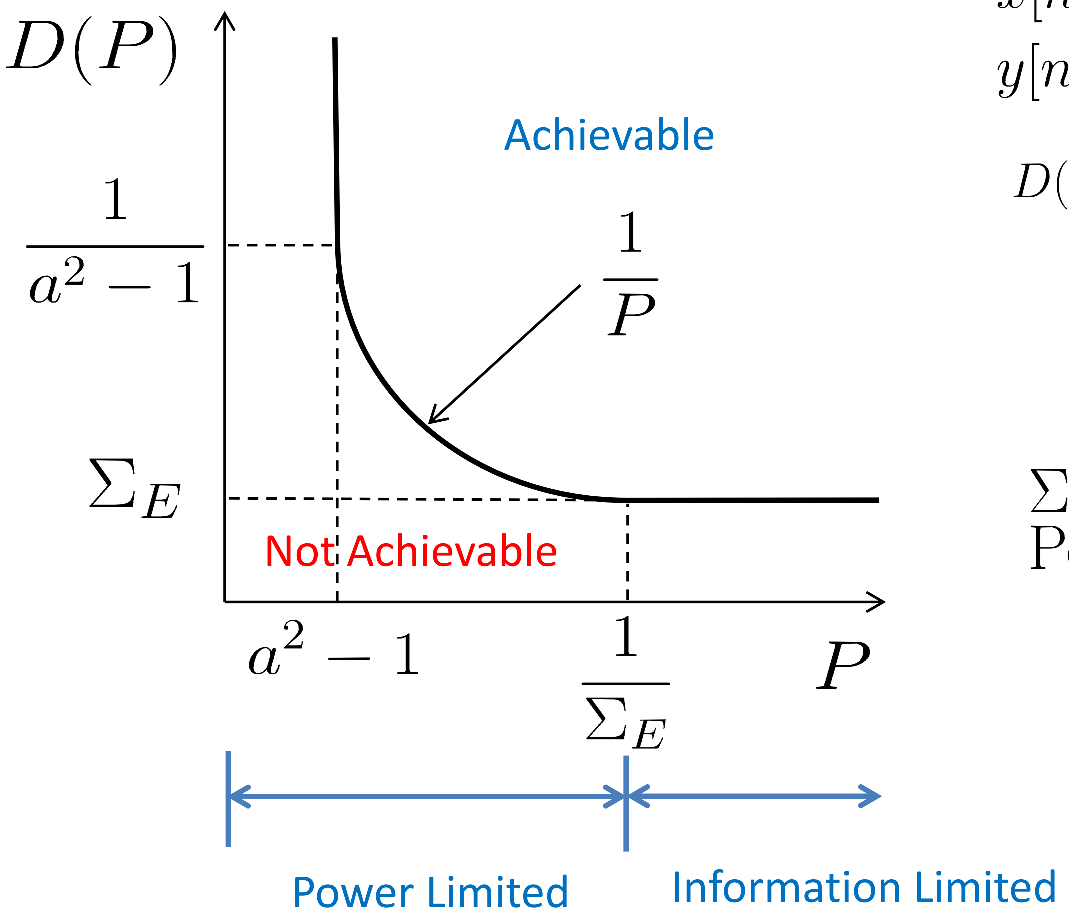
$$\inf_u \frac{1}{N} \sum_{0 \le n < N} \mathbb{E}[qx^2[n] + ru^2[n]]$$

Power-Distortion Tradeoff:

$$D(P) = \inf_u \frac{1}{N} \sum_{0 \le n < N} \mathbb{E}[x^2[n]]$$
$$\text{s.t. } \frac{1}{N} \sum_{0 \le n < N} \mathbb{E}[u^2[n]] \le P$$

Power-Distortion Tradeoff: When 1<|a|<4

Conceptual Picture of the Tradeoff Curve



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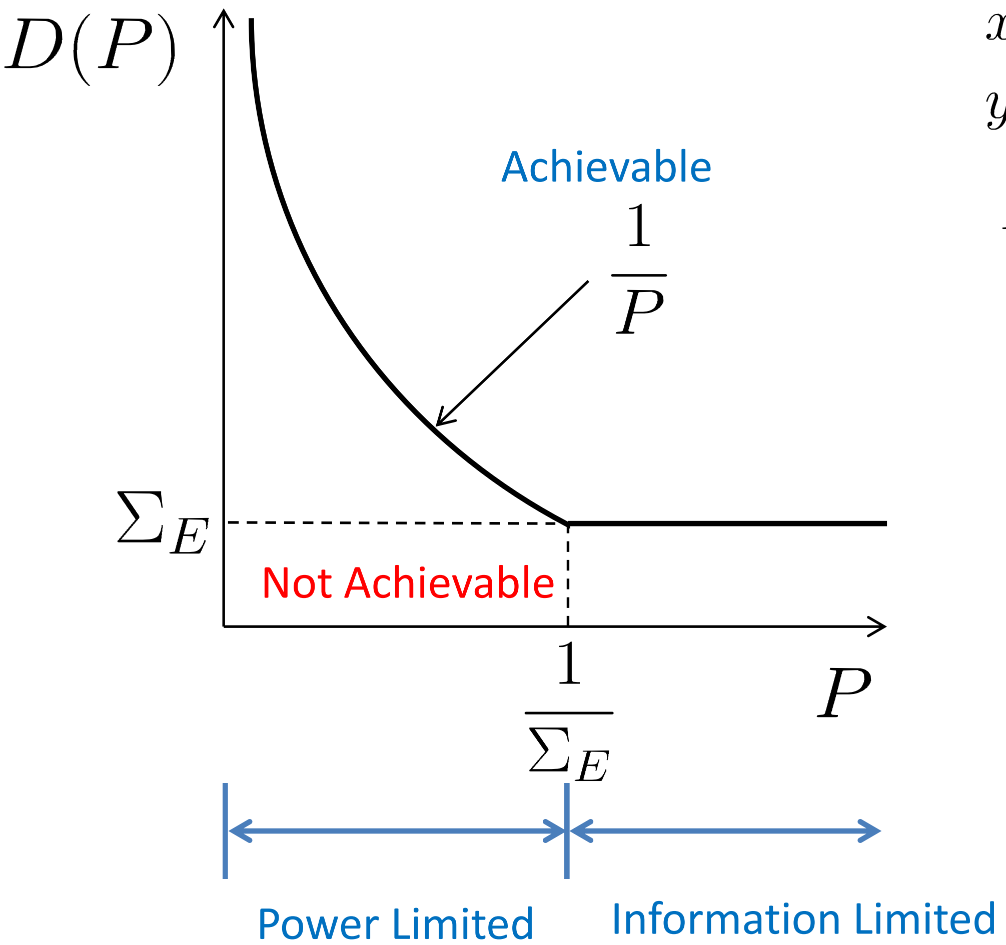
Σ_E: Kalman Filter Performance

Power-Distortion Tradeoff: When |a|=1

Conceptual Picture of the Tradeoff Curve



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Σ_E: Kalman Filter Performance

Power-Distortion Tradeoff: When |a|<1

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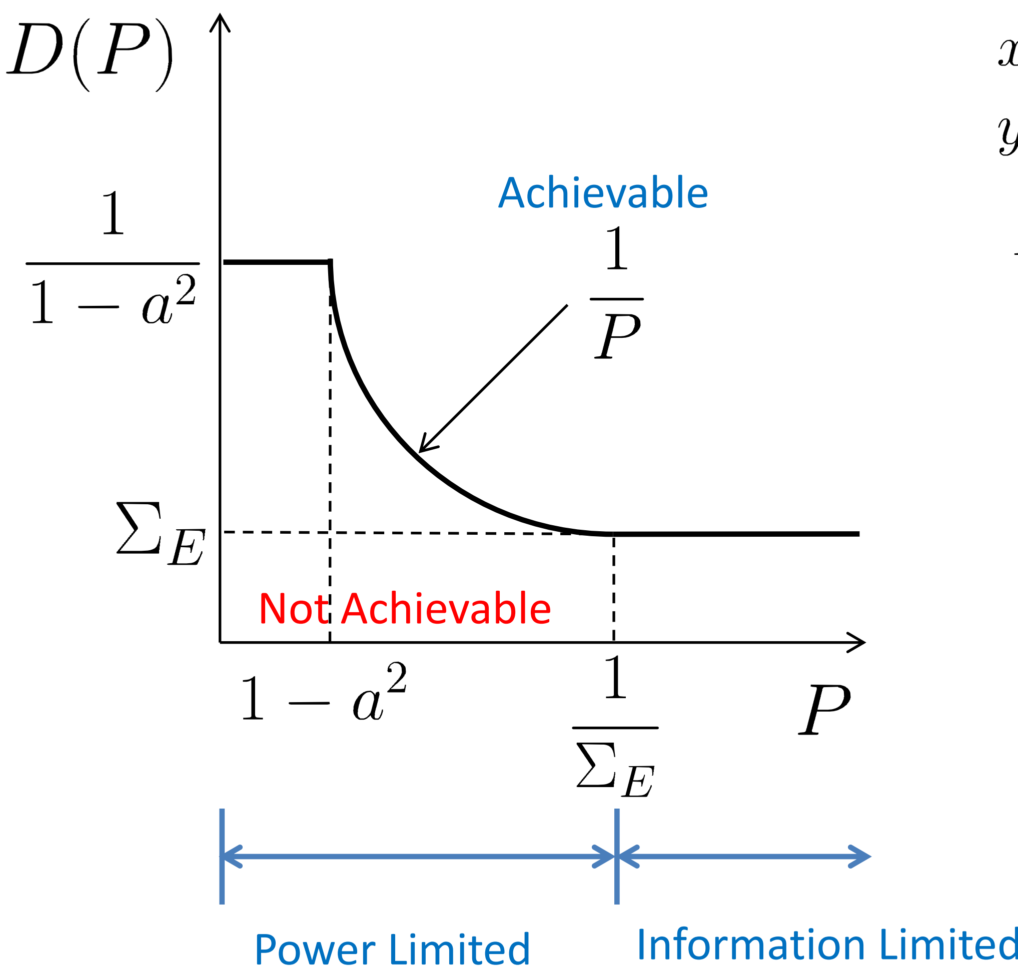

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Σ_E: Kalman Filter Performance

Approximately Optimal Strategy

Linear Strategy L_{lin}

$$u_1[n] = 0$$
$$u_2[n] = -k\mathbb{E}[x[n]|y_2^n]$$

or

$$u_1[n] = -k\mathbb{E}[x[n]|y_1^n]$$
$$u_2[n] = 0$$

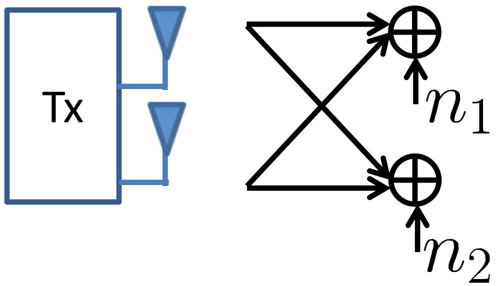
Theorem [Park and Sahai, 2012]

There exists $c < 6 \cdot 10^6$ such that for all $|a| < 4$, q, r_1, r_2, σ_w, σ_{v1}, σ_{v2}

$$\frac{\inf_{u_1,u_2 \in L_{lin}} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1u_1^2[n] + r_2u_2^2[n]]}{\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1u_1^2[n] + r_2u_2^2[n]]} \le c$$

Conclusion

MIMO Communication Problem



Decentralized Scalar LQG Problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n] + v_1[n]$$
$$y_2[n] = x[n] + v_2[n]$$

Linear Super-position of Signals, Gaussian Disturbance

Divide Cases:

(1) High-SNR (Signal-to-Noise Ratio)
- d.o.f. gain (rank of signal) is important
- Rank maximization scheme

(1) Low-SNR (Signal-to-Noise Ratio)
- Beam-forming gain (power of signal) is important
- Maximum-Ratio combining scheme

Divide Cases:
How we get the Information for control?

(1) Fast Dynamics
- Implicit Communication from the other controller

(2) Slow Dynamics
- Kalman filtering gain is crucial

Conclusion

Centralized LQG Problem

$$x[n+1] = Ax[n] + Bu[n] + w[n]$$
$$y[n] = Cx[n] + v[n]$$

- Finite-Dimensional Solution
- Estimation-Control Separation

Decentralized Scalar LQG Problem

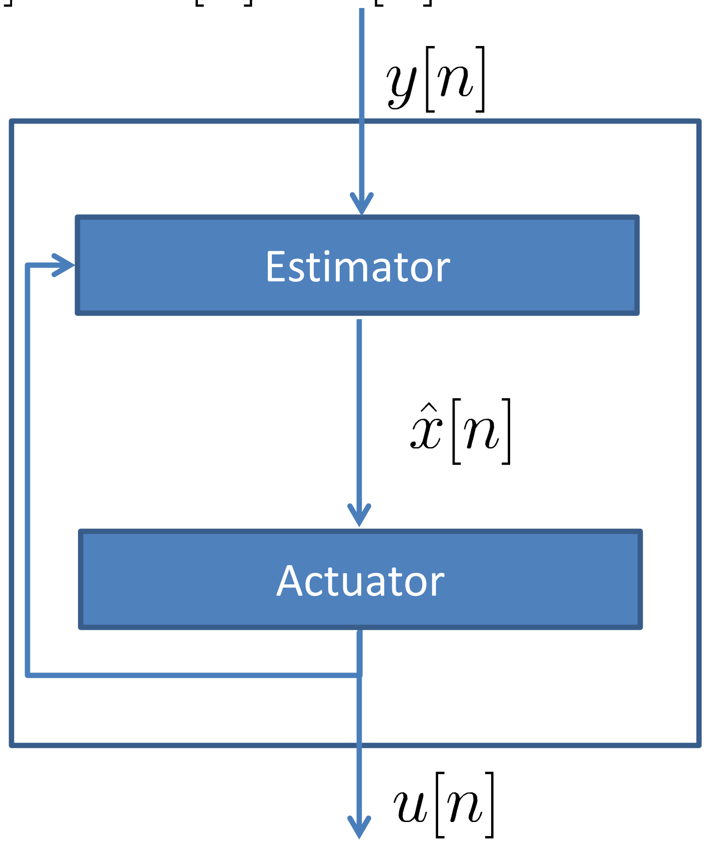
$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n] + v_1[n]$$
$$y_2[n] = x[n] + v_2[n]$$

- Finite-Dimensional Approximated Solution
- Estimation-Control Separation and Implicit Communication Strategy

Conjecture

Centralized LQG Problem

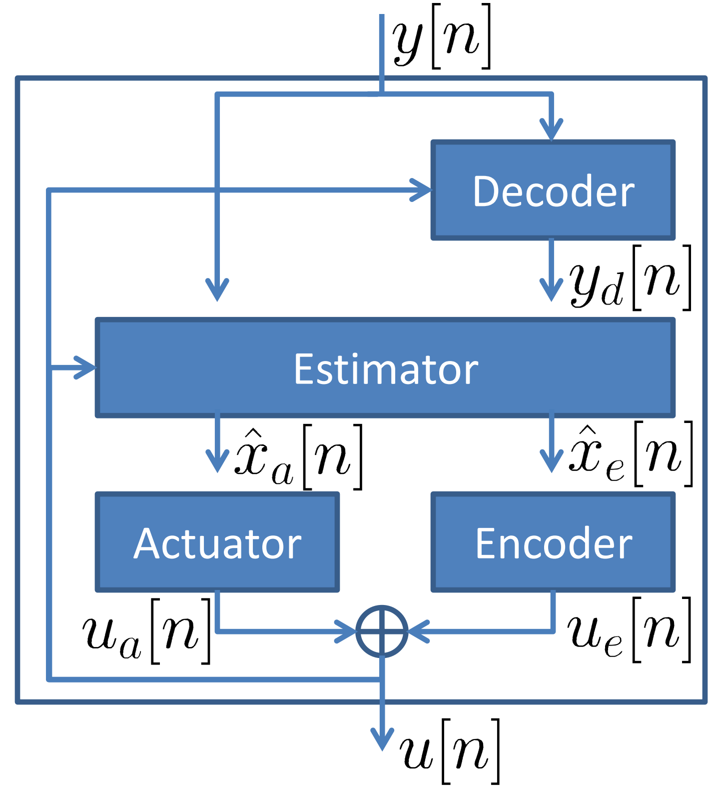
$$x[n+1] = Ax[n] + Bu[n] + w[n]$$
$$y[n] = Cx[n] + v[n]$$



Estimation-Control Separation

Decentralized LQG Problem

$$x[n+1] = Ax[n] + B_1u_1[n] + \cdots + B_mu_m[n] + w[n]$$
$$y_1[n] = C_1x[n] + v_1[n]$$
$$\vdots$$
$$y_m[n] = C_mx[n] + v_m[n]$$



Communication-Estimation-Control Separation

- Thank you