

On the capacity regions of broadcast channel problems with receiver side information

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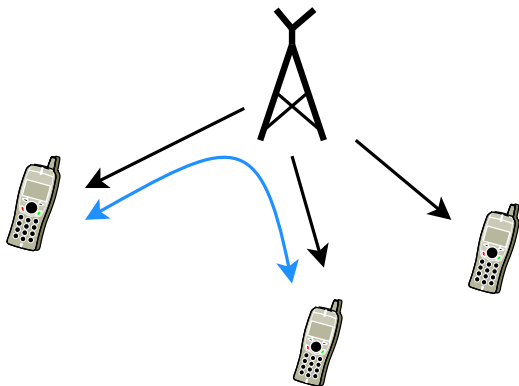
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Motivation: In-cell Communication

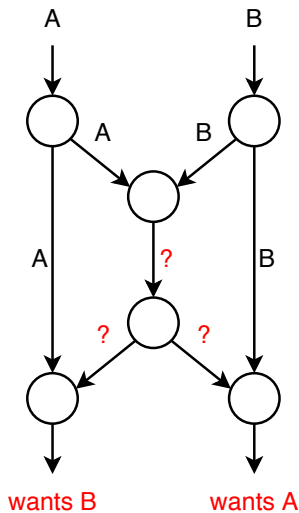


- Two nodes want to communicate within a cell.
- Provider does not allow direct communication.
 - Uplink: Messages send & decoded at basestation (MAC).
 - **Advanced downlink:** (Partial) messages cognition at receiver.

Outline

- 1 Introduction
 - Motivation
 - Basic Idea of Network Coding
 - Capacity Region of the Bidirectional Broadcast Channel
 - Vector-valued Gaussian case
- 2 Bidirectional BC with Random States
 - Achievable Region
 - Capacity Region if State is also known at one Receiver
 - Gaussian Channel
- 3 Three User Broadcast Channel with Message Cognition
 - Coding Strategy
 - Capacity Results for Special Cases
 - Look into a Converse for Less Noisy
- 4 Concluding Remarks

Basic Idea of Network Coding

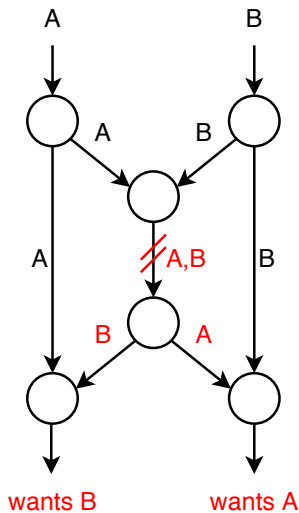


Example: **Butterfly Network**

- Problem: Bits A and B should be transferred.
- Constraint: Each P2P link has 1 bit capacity.

- Network coding: Ahlswede et al, "Network Information Flow" T-IT 00.

Basic Idea of Network Coding

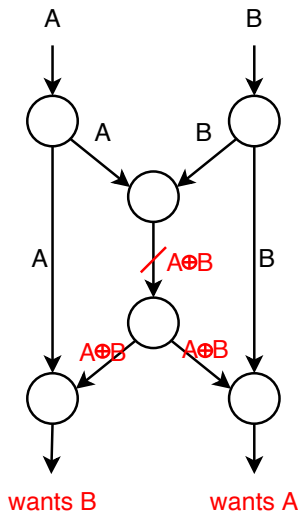


Example: **Butterfly Network**

- Problem: Bits A and B should be transferred.
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- Routing: Forward A and B
⇒ **Two channel uses!**

- Network coding: Ahlswede et al, "Network Information Flow" T-IT 00.

Basic Idea of Network Coding



Example: **Butterfly Network**

- Problem: Bits A and B should be transferred.
- Constraint: Each P2P link has 1 bit capacity.
- **Routing:** Forward A and B
⇒ **Two channel uses!**
- **Network coding:** Forward $A \oplus B$
⇒ **One channel use!**

Idea: Allow computation at nodes!

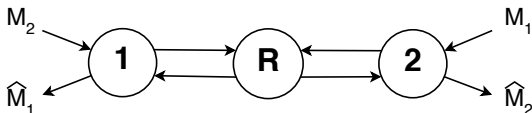
Paradigm Shift

Information flows \neq incompressible fluids!

- Network coding: Ahlswede et al, "Network Information Flow" T-IT 00.

Bidirectional Relaying

- Two nodes want to **exchange messages** with the help of a relay.
 - For scenarios where the direct link is not good enough!
 - *Half-duplex assumption*: Nodes can either transmit or receive.

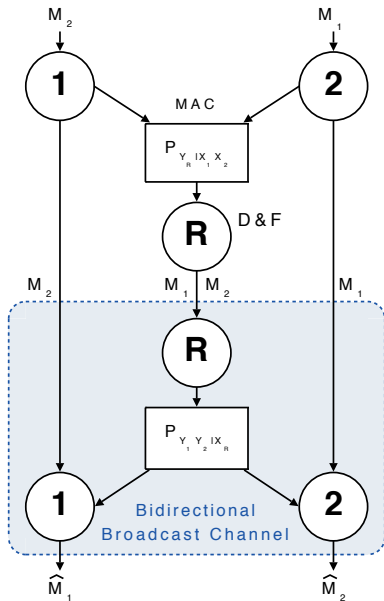


- Different **processing strategies** at relay node:
 - amplify-and-forward, decode-and-forward (here), compress-and-forward, compute-and-forward, ...
 - **optimal strategy unknown**

Bidirectional Broadcast Channel

Restricted decode & forward bidirectional relaying

1. Phase: MAC [Ahlsvede '71]
2. Phase: BiBC: BC with RX message cognition



Bidirectional Broadcast Channel

DM-BiBC capacity region [T-IT '08]

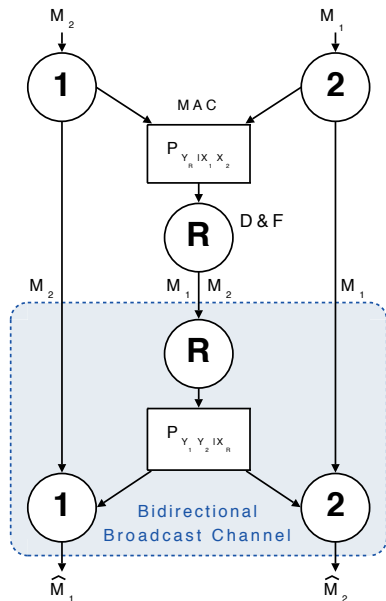
Union of all $[R_1, R_2]$ over p_{X_R} :

$$0 \leq R_1 \leq I(X_R; Y_1)$$

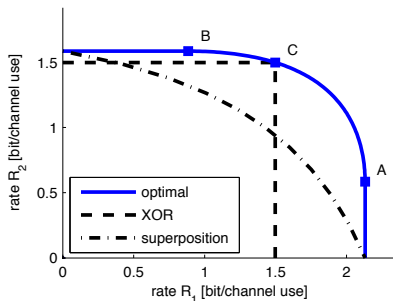
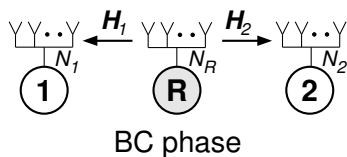
$$0 \leq R_2 \leq I(X_R; Y_2)$$

Proof ideas:

- Coding which combines information flows at relay (network coding idea).
- Converse: Take side information into account.



Gaussian Multi-Antenna Bidirectional Relaying



Capacity Region [ISIT '08]

$$C_{\text{BC}} := \bigcup_{\substack{\text{tr } \mathbf{Q} \leq P, \\ \mathbf{Q} \geq 0}} \{ [R_1, R_2] \in \mathbb{R}_+^2 : R_1 \leq C_1(\mathbf{Q}), R_2 \leq C_2(\mathbf{Q}) \}$$

with

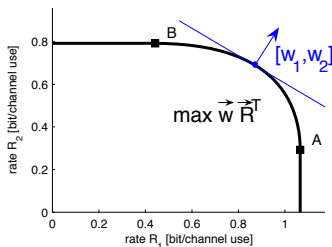
$$C_i(\mathbf{Q}) := \log \det \left(\mathbf{I}_{N_i} + \frac{1}{\sigma^2} \mathbf{H}_i^H \mathbf{Q} \mathbf{H}_i \right), \quad i = 1, 2.$$

Transmit Covariance Optimization Problem

Boundary characterized by

$$\arg \max_{\text{tr } \mathbf{Q} \leq P, \mathbf{Q} \geq 0} \sum_{i=1}^2 w_i \log \det \left(\mathbf{I}_{N_i} + \frac{1}{\sigma^2} \mathbf{H}_i^H \mathbf{Q} \mathbf{H}_i \right)$$

▣▣▣▣ **Convex opt. problem!**



Closed form results/procedures are available:

- **MISO case** [T-SP '09]
 - Rank one optimality of transmit covariance matrix \mathbf{Q} .
- **MIMO case** [T-COM '09]
 - Generalized water-filling solution for high SNR and non-degenerate channels ($(\mathbf{H}_i \mathbf{H}_i^H)^{-1}$ exists)

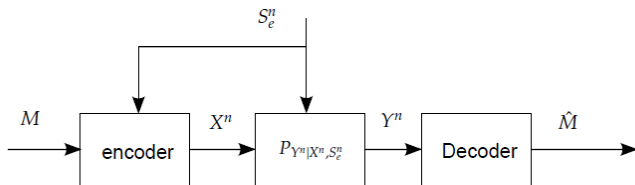
Lesson learned so far

Paradigm shift

Information flow \neq fluids.

- **Communication principle:** Convey as much information to the receiving nodes which allows them to conclude on the message using their side information.
- ⇒ **Bidirectional broadcast channel:**
Single information flow used by both users.
- **Trade-off:** Optimal input distribution need not be optimal for both users (vector optimization problem).

Channel Coding with States

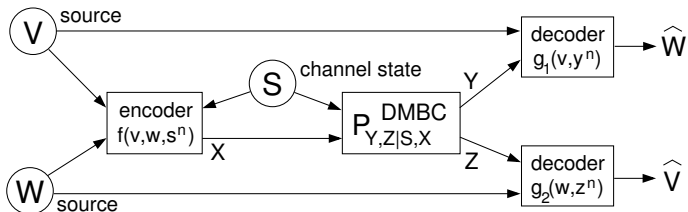


- Channel coding with states known non-causally at the encoder [Gel'fand, Pinsker '80]

$$C = \max_{P_{X|U,S} P_{U|S}} [I(U; Y) - I(U; S)] \quad (\geq \max_{P_{X|U,S} P_U} I(U; Y))$$

- Extension to broadcast channel with state [Steinberg '05] and [Steinberg, Shamai '05]
 - Capacity result for degraded case.
 - General case open - similar results as for BC

Bidirectional DMBC with Random States



Definition: DMBC with random states

$$P_{Y^n Z^n | S^n X^n}(y^n, z^n | s^n, x^n) = \prod_{i=1}^n P_{YZ|SX}(y_i, z_i | s_i, x_i), \quad P_S^n(s^n) = \prod_{i=1}^n P_S(s_i)$$

Achievable Rate Region

- Extension of Gel'fand Pinsker coding results for BC with receiver side information (bidirectional BC) [accepted T-IT]:

Achievable Rate Region

Convex hull of the set of all rate pairs $[R_1, R_2]$ such that

$$R_1 \leq [I(U; Y) - I(U; S)]_+, \quad R_2 \leq [I(U; Z) - I(U; S)]_+$$

for some $U - (X, S) - (Y, Z)$ with $p_{X|US}$ deterministic and $|\mathcal{U}| \leq |\mathcal{S}||\mathcal{X}| + 1$ sufficient.

Trivial Outer Bound

Set of all rate pairs $[R_1, R_2]$ such that

$$R_1 \leq \max_{P_{U,X|S}} [I(U; Y) - I(U; S)], \quad R_2 \leq \max_{P_{U,X|S}} [I(U; Z) - I(U; S)]$$

Sketch of coding scheme

- **Random codebook:** Generate $2^{n(R_1+R_2+\tilde{R})}$ iid sequences $u^n(v, w, \ell) \sim P_U$, $1 \leq v \leq 2^{nR_1}$, $1 \leq w \leq 2^{nR_2}$, $1 \leq \ell \leq 2^{n\tilde{R}}$.
- **Encoding:** To send (v, w) after observing s^n look for some $\ell : (u^n(v, w, \ell), s^n) \in \mathcal{T}_\varepsilon^{(n)}(P_{US})$.
 - Probability of success tends to one with n if $\tilde{R} > I(U; S)$.
- **Decoding:** Decoder g_1 knows v and searches for a unique pair $(\hat{w}, \hat{\ell})$ such that $(u^n(v, \hat{w}, \hat{\ell}), y^n) \in \mathcal{T}_\varepsilon^{(n)}(P_{UY})$.
 - Probability of failure vanishes with n if $R_1 + \tilde{R} < I(U; Y)$.Likewise, decoder g_2 knows w and searches for a unique pair $(\hat{v}, \hat{\ell})$ such that $(u^n(\hat{v}, w, \hat{\ell}), z^n) \in \mathcal{T}_\varepsilon^{(n)}(P_{UZ})$.
 - Probability of failure vanishes with n if $R_2 + \tilde{R} < I(U; Z)$.

Channel State additionally known at Decoder one

Capacity Region

Set of all rate pairs $[R_1, R_2]$ such that

$$R_1 \leq I(X; Y|S), \quad R_2 \leq I(U; Z) - I(U; S)$$

for some $U - (X, S) - (Y, Z)$ with $p_{X|US}$ deterministic.

- **Equivalent representation of region (crucial):**

- Consider output $\tilde{Y} = (Y, S)$

$$I(\tilde{Y}; U) - I(S; U) = I(Y, S; U) - I(S; U) = I(Y; U|S)$$

- Since $p_{X|US}$ deterministic $\Rightarrow X - (U, S) - Y$, we have

$$I(U; Y|S) = I(X, U; Y|S) - \underbrace{I(X; Y|U, S)}_{=0, X-(U,S)-Y} = I(X; Y|S) + \underbrace{I(U; Y|X, S)}_{=0, U-(X,S)-Y} = I(X; Y|S)$$

- Previous gives achievability; standard arguments the converse

Scalar Complex Gaussian Channel

$$Y = X + S + N_1, \quad Z = X + S + N_2$$

- Power constraint $\mathbb{E}\{X^2\} \leq P$
- State $S \sim \mathcal{CN}(0, Q)$ and channel noise $N_i \sim \mathcal{CN}(0, \sigma_i^2)$, $i = 1, 2$.

Costa's choice of RV $U = X + \alpha S$, $X \sim \mathcal{CN}(0, Q)$, $X \perp S$:

$$R_i(\alpha) = \log \left(\frac{P(P + Q + \sigma_i^2)}{PQ(1 - \alpha)^2 + \sigma_i^2(P + \alpha^2 Q)} \right),$$

maximized at $\alpha_i^* = P/(P + \sigma_i^2)$, $i = 1, 2$.

- *Remark:* If $\sigma_1^2 \neq \sigma_2^2$ then $\alpha_1^* \neq \alpha_2^*$...

Channel State additionally known at Decoder one

Capacity region

Set of rate pairs $[R_1, R_2]$ such that

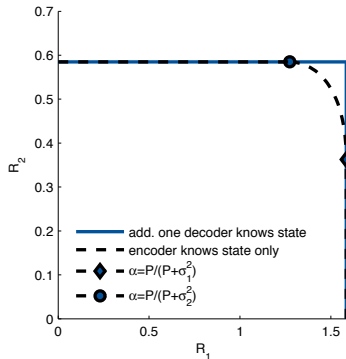
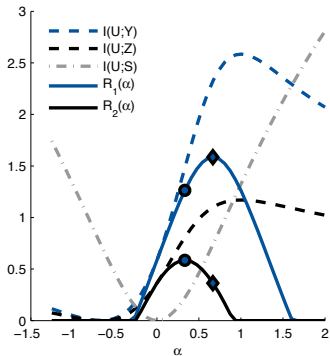
$$\begin{aligned}R_1 &\leq R_1(\alpha_2^*) = I(U; X + S + N_1|S) \\ &= I(X + \alpha_2^* S; X + N_1|S) \\ &= I(X; X + N_1) = \log\left(1 + P/\sigma_1^2\right),\end{aligned}$$

$$\begin{aligned}R_2 &\leq R_2(\alpha_2^*) = I(X; X + S + N_2|S) \\ &= I(X; X + N_2) = \log\left(1 + P/\sigma_2^2\right).\end{aligned}$$

On each link the AWGN single-user capacity can be achieved.

⇒ Capacity region

Illustration



- If only the encoder knows channel state sequence, than each single-user capacity is achievable, but not simultaneously.
- If additionally one decoder knows the channel state sequence, both user can simultaneously achieve single-user capacity

Looking backward, looking forward

Backward:

- 2-user broadcast channel with receiver message cognition
- 2-user broadcast channel with receiver message cognition and random state

Forward:

- 3-user broadcast channel with partial message cognition and degraded message sets

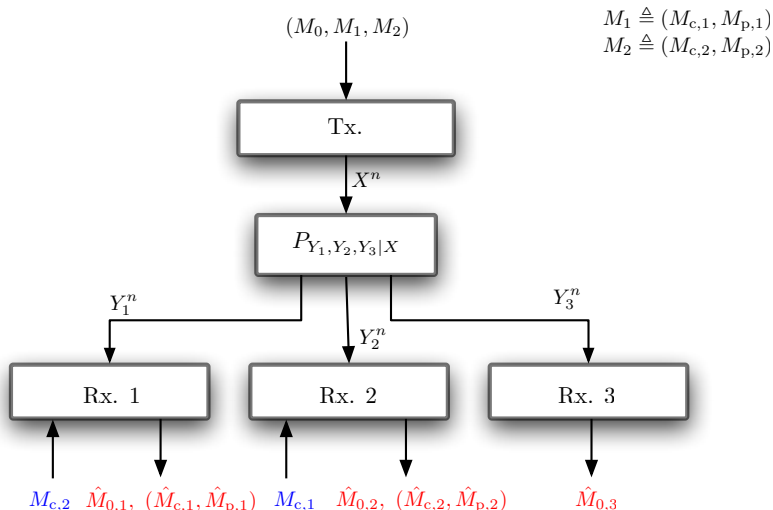
Three User Extension

- **Capacity results** are found for the **general 2 receiver BC** with
 - full message cognition, called *bidirectional broadcast channel* [T-IT '08]
 - partial message cognition and degraded message set [Kramer, Shamai, '07]
 - degraded message sets [Körner, Marton, '77]
- Some results on the capacity for **3 receiver BC**, degraded message set, no message cognition [Nair, El Gamal, '09]

Question: Can we obtain capacity results for the 3 receiver BC, degraded message set, and full/partial message cognition?

Answer: For special cases/classes only.

Problem: BC with Partial Message Cognition



- Problem includes the general 2 receiver BC!

Broadcast Channel Class: Less Noisy

Definition [Körner, Marton, '75]

- Y is **less noisy** than Z (notation: $Y \succeq Z$) if

$$I(U; Y) \geq I(U; Z) \quad \text{for all } U - X - (Y, Z).$$

Capacity result for

- 2 receiver less noisy BC [Körner, Marton, '75]
- 3 receiver less noisy BC [Nair, Wang '11]

Key lemma [Nair, Wang, '11]

Let $X \rightarrow (Y, Z)$ be DM-BC with $Y \succeq Z$ and M any RV such that $M - X^n - (Y^n, Z^n)$, then

- 1 $I(Y^{i-1}; Z_i | M) \geq I(Z^{i-1}; Z_i | M), \quad 1 \leq i \leq n.$
- 2 $I(Y^{i-1}; Y_i | M) \geq I(Z^{i-1}; Y_i | M), \quad 1 \leq i \leq n.$

Inner Bound

- $\mathcal{R}_{\text{in,part}}^{(1)}$ denotes the set of $(R_0, R_{1,c}, R_{1,p}, R_{2,c}, R_{2,p})$ satisfying

$$R_0 \leq I(U; Y_3)$$

$$R_{1,p} \leq I(X; Y_1|V)$$

$$R_0 + R_2 \leq \min\{I(V; Y_2), I(U; Y_3) + I(V; Y_2|U)\}$$

$$R_0 + R_1 + R_{2,p} \leq \min\{I(X; Y_1), I(U; Y_3) + I(X; Y_1|U)\}$$

for some (U, V, X) with $U - V - X - (Y_1, Y_2, Y_3)$.

- $\mathcal{R}_{\text{in,part}}^{(2)}$: Interchange indices 1 and 2.

- $|\mathcal{U}| \leq |\mathcal{X}| + 4$ and $|\mathcal{V}| \leq (|\mathcal{X}| + 4)(|\mathcal{X}| + 1)$ suffices.

Theorem: Achievable Rate Region $\mathcal{R}_{\text{in,part}}$ [ISIT'12]

$$\mathcal{C}_{\text{part}} \supseteq \mathcal{R}_{\text{in,part}} \triangleq \text{convex hull}(\mathcal{R}_{\text{in,part}}^{(1)} \cup \mathcal{R}_{\text{in,part}}^{(2)})$$

Proof of $\mathcal{R}_{\text{in,part}}^{(1)}$: Superposition Coding

- **Rate splitting and construct new messages**

- private message $M_{2,p} = (M_{2,p}^{(1)}, M_{2,p}^{(2)})$

- cognizant messages: $M_{1,c} = (M_{1,c}^{(1)}, M_{1,c}^{(2)})$, $M_{2,c} = (M_{2,c}^{(1)}, M_{2,c}^{(2)})$

$$M_{\oplus}^{(k)} = (M_{1,c}^{(k)} + M_{2,c}^{(k)}) \text{ modulo } 2^{n \max\{R_{1,c}^{(k)}, R_{2,c}^{(k)}\}}, \quad k = 1, 2$$

Rx1: Decide on $M_{1,c}^{(k)}$ using knowledge of $M_{2,c}^{(k)}$

Rx2: Decide on $M_{2,c}^{(k)}$ using knowledge of $M_{1,c}^{(k)}$

Proof of $\mathcal{R}_{\text{in,part}}^{(1)}$: Superposition Coding

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Rx1: Decide on $M_{1,c}^{(k)}$ using knowledge of $M_{2,c}^{(k)}$

Rx2: Decide on $M_{2,c}^{(k)}$ using knowledge of $M_{1,c}^{(k)}$

- **3 layer superposition coding with non-unique decoding:**

layer	codeword	M_0	$M_{\oplus}^{(1)}$	$M_{2,p}^{(1)}$	$M_{\oplus}^{(2)}$	$M_{2,p}^{(2)}$	$M_{1,p}$	used by
1	u^n iid $\sim P_U$	×	×	×				nodes 1,2,3
2	v^n iid $\sim P_{V U}$	×	×	×	×	×		nodes 1,2
3	x^n iid $\sim P_{X V}$	×	×	×	×	×	×	node 1

□

Less Noisy Capacity Results

Theorem: Capacity Region [ISIT'12]

If there is a *less-noisy ordering* between Y_1 , Y_2 , and Y_3 , then

$$C_{\text{part}} = \mathcal{R}_{\text{in,part}}.$$

In particular the description of $\mathcal{R}_{\text{in,part}}$ can be simplified, e.g.

$$\bullet \quad Y_1 \succeq Y_2 \succeq Y_3 \quad \Rightarrow \quad \begin{aligned} I(V; Y_2) &\geq I(U; Y_3) + I(V; Y_2|U) \\ I(X; Y_1) &\geq I(U; Y_3) + I(X; Y_1|U) \end{aligned}$$

Converse is proved for the following region, which includes the simplified achievable rate region.

$$R_0 \leq I(U; Y_3)$$

$$R_{1,p} \leq I(X; Y_1|V)$$

$$R_2 \leq I(V; Y_2|U)$$

$$R_1 + R_{2,p} \leq I(X; Y_1|U)$$

Example: Gaussian Channel

$$Y_k = X + N_k, \quad N_k \sim \mathcal{N}(0, \sigma_k^2), \quad k = 1, 2, 3$$

Corollary: Capacity Region Gaussian Channel

If $\sigma_3^2 \geq \sigma_2^2 \geq \sigma_1^2$, then C_{part} is given by

$$R_0 \leq \frac{1}{2} \log \left(1 + \frac{\alpha P}{(1-\alpha)P + \sigma_3^2} \right)$$

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{(1-\alpha-\beta)P}{\sigma_1^2} \right) + R_{1,c}$$

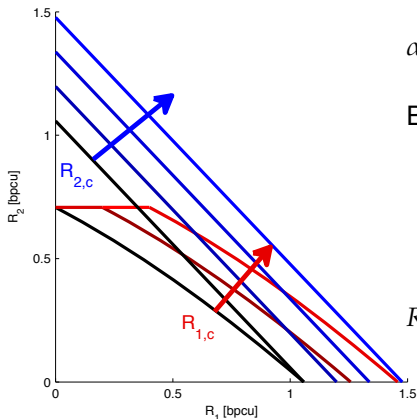
$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{\beta P}{(1-\alpha-\beta)P + \sigma_2^2} \right)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{(1-\alpha)P}{\sigma_1^2} \right) + R_{2,c} \quad \alpha, \beta, \alpha + \beta \in [0, 1]$$

Proof: Entropy power inequality & maximal entropy property.

Discussion: Gaussian Channel $\sigma_3^2 \geq \sigma_2^2 \geq \sigma_1^2$

Cognizant knowledge: $R_{2,c} \rightarrow \text{Rx 1}$, $R_{1,c} \rightarrow \text{Rx 2}$



$$\alpha_0: R_0 = \frac{1}{2} \log \left(1 + \frac{\alpha_0 P}{(1-\alpha_0)P + \sigma_3^2} \right)$$

Bounds on R_1 and R_2 , $\beta \in [0, \alpha_0]$:

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{(1-\alpha_0-\beta)P}{\sigma_1^2} \right) + R_{1,c}$$

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{\beta P}{(1-\alpha_0-\beta)P + \sigma_2^2} \right)$$

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{(1-\alpha_0)P}{\sigma_1^2} \right) + R_{2,c}$$

\Rightarrow There might be no gain due to (more) message cognition!

Full Message Cognition: $M_{1,c} = M_1$ and $M_{2,c} = M_2$

- Two examples for which capacity is known. (3 more in [ISIT'12])
 - Cases which we cannot solve without full message cognition.

Theorem: Capacity Region

The capacity region for the full message cognition case if

(ii) Y_3 is *more capable* than Y_1 and Y_2 :

$$R_0 + R_1 \leq I(X; Y_1)$$

$$R_0 + R_2 \leq I(X; Y_2)$$

(v) Y_3 is a *deterministic function* of X :

$$R_0 \leq H(Y_3)$$

$$R_0 + R_1 \leq I(X; Y_1)$$

$$R_0 + R_2 \leq I(X; Y_2)$$

- Main task: Simplify achievable rate region. Converses are easy.

Look into a Converse for $Y_1 \geq Y_2 \geq Y_3$

$$\begin{aligned}
 n(R_1 + R_{2,p}) - n\epsilon_n &\stackrel{\text{Fano}}{\leq} \underbrace{I(M_1; Y_1^n | M_0, M_{2,c})}_{\substack{= \sum_{i=1}^n I(M_1, Y_{2,i+1}^n; Y_{1,i} | M_0, M_{2,c}, Y_1^{i-1}) \\ - I(Y_{2,i+1}^n; Y_{1,i} | M_0, M_1, M_{2,c}, Y_1^{i-1})}} + \underbrace{I(M_{2,p}; Y_2^n | M_0, M_1, M_{2,c})}_{\substack{= \sum_{i=1}^n I(M_{2,p}; Y_{2,i} | M_0, M_1, M_{2,c}, Y_{2,i+1}^n, Y_1^{i-1}) \\ + I(Y_1^{i-1}; Y_{2,i} | M_0, M_1, M_{2,c}, Y_{2,i+1}^n)}} \\
 &\stackrel{\text{C.S.}}{\leq} \sum_{i=1}^n \underbrace{I(M_1, M_{2,c}, Y_{2,i+1}^n; Y_{1,i} | M_0, Y_1^{i-1}) + I(X_i; Y_{2,i} | M_0, M_1, M_{2,c}, Y_{2,i+1}^n, Y_1^{i-1})}_{\substack{Y_1 \geq Y_2 \\ \leq I(X_i; Y_{1,i} | M_0, M_1, M_{2,c}, Y_{2,i+1}^n, Y_1^{i-1})}} \\
 &\leq \sum_{i=1}^n I(X_i; Y_{1,i} | M_0, Y_1^{i-1}) = \sum_{i=1}^n I(X_i; Y_{1,i} | M_0) - \underbrace{I(Y_{1,i}; Y_1^{i-1} | M_0)}_{\substack{\text{N.W. Lemma} \\ \leq I(Y_{1,i}; Y_2^{i-1} | M_0)}} \\
 &\leq \sum_{i=1}^n I(X_i; Y_{1,i} | Y_2^{i-1}, M_0) = \sum_{i=1}^n I(X_i; Y_{1,i} | U_i)
 \end{aligned}$$

using $(Y_{1,i}, Y_{2,i}) - X_i - (M_0, M_1, M_2, Y_1^{i-1}, Y_{2,i+1}^n)$ and $(Y_1^n, Y_2^n) - X^n - M_0$.

Concluding Remarks

- Capacity for general bidirectional BC is known, but extension to general 3 receiver BC with **full receiver message cognition** and **degraded message sets** appears to be difficult.
 - Problem: Extension of Csiszar sum lemma.
- *Observation:* (More) receiver message cognition might not enlarge capacity region.
 - RX cognition approach useful for genie aided converses?
- Broadcast with (partial) message cognition relevant for
 - cellular communication
 - file-exchange problems

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Thank you for your attention! Questions?