A Mean Field Games Approach to Consensus Problems

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Outline

- Background:
  - Mean Field Game (MFG) Theory
  - Standard Consensus Algorithms (SCAs)
- MFG Consensus Formulation and Solution – Homogenous Case
- MFG Consensus Formulation and Solution – Heterogeneous Case

Background – Mean Field Game (MFG) Theory

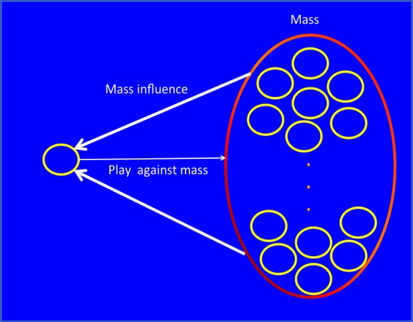
The Modeling Setup of Mean Field Game Theory (Huang, Caines, Malhamé ('03,'06,'07), Lasry-Lions ('06,'07)):

- For a class of dynamic games with a large number of minor agents
- Each minor agent interacts with the average or so-called mass effect of other agents via couplings in their individual cost functions and individual dynamics
- A minor agent is an agent which, asymptotically as the population size goes to infinity, has a negligible influence on the overall system while the overall population's effect on it is significant

Background – Mean Field Game (MFG) Theory

Key Idea of Mean Field Game (MFG) Theory (HCM ('03,'06,'07)):

- Establish the existence of an equilibrium relationship between the individual strategies and the mass effect in the infinite population limit
- Such that the individual strategy of each agent is a best response to the mass effect, and the set of the strategies collectively replicate that mass effect
- Apply the resulting infinite population strategies to a finite population system and obtain suitable approximate equilibrium



Background – MFG-LQG Problem Formulation

Basic Linear-Qudratic-Gaussian (LQG) Dynamic Game Problem

- Individual Agent's Dynamics:

$$dz_i(t) = \big(a_i z_i(t) + b u_i(t)\big)dt + \sigma_i dw_i(t), \quad 1 \le i \le N$$

N: population size, z_i: state of agent i, u_i: control input, w_i: disturbance

- Individual Agent's Cost Function:

$$J_i(u_i, \nu) \triangleq E \int_0^\infty e^{-\rho t} \Big(\big(z_i(t) - \nu(t)\big)^2 + r u_i^2(t) \Big) dt$$

$\rho > 0$: discount factor, $r > 0$: control penalty, and

$$\nu(\cdot) \triangleq \gamma\big(\frac{1}{N} \sum_{k=1}^{N} z_k(\cdot) + \eta\big)$$

Main feature:

- Agents are coupled via their costs
- Stochastic tracked process ν:
  - (i) depends on other agents' control laws
  - (ii) not feasible for z_i to track all z_k trajectories for large N

Background – Preliminary LQG Tracking Problem

Preliminary LQG Tracking Problem For One Agent Only: $x^*(\cdot)$ known and deterministic

$$dz_i(t) = \big(a_i z_i(t) + bu_i(t)\big)dt + \sigma_i dw_i(t)$$

$$J_i(u_i, x^*) = E\int_0^\infty e^{-\rho t}\Big(\big(z_i(t) - x^*(t)\big)^2 + ru_i^2(t)\Big)dt$$

Computation of the Optimal Tracking Control:

$$u_i(\cdot) = -\frac{b}{r}\big(\Pi_i z_i(\cdot) + s_i(\cdot)\big)$$

Riccati Equation: $\rho\Pi_i = 2a_i\Pi_i - \frac{b^2}{r}\Pi_i^2 + 1, \quad \Pi_i > 0$

Mass Offset Control: $-\frac{ds_i}{dt} = -\rho s_i + a_i s_i - \frac{b^2}{r}\Pi_i s_i - x^*$

- Boundedness condition on $x^*(\cdot)$ implies existence of unique solution s_i

Background – The Fundamental MFG-LQG System

Continuum of Systems under Optimal LQG Tracking Control:
$a \in \mathcal{A}$; common b for simplicity

$$-\frac{ds_a}{dt} = -\rho s_a + a s_a - \frac{b^2}{r}\Pi_a s_a - x^* \quad \text{(Tracking mass equation)}$$

$$\frac{d\bar{z}_a}{dt} = (a - \frac{b^2}{r}\Pi_a)\bar{z}_a - \frac{b^2}{r} s_a \quad \text{(The mean state equation)}$$

$$\bar{z}(t) = \int_{\mathcal{A}} \bar{z}_a(t) dF(a) \quad \text{(The mean field function)}$$

$$x^*(t) = \gamma(\bar{z}(t) + \eta) \qquad t \geq 0 \quad \text{(The mass function)}$$

$$\text{Riccati Equation}: \quad \rho\Pi_a = 2a\Pi_a - \frac{b^2}{r}\Pi_a^2 + 1, \quad \Pi_a > 0$$

- $F(\cdot)$: The limit empirical distribution of $\{a_i : i > 1\} \subset \mathcal{A}$
- Individual control action $u_a = -\frac{b}{r}(\Pi_a z_a + s_a)$ is optimal w.r.t tracked x^*
- Does there exist a solution $(\bar{z}_a, s_a, x^*; a \in \mathcal{A})$? Yes: Fixed Point Theorem

Background – Properties of MFG-LQG Solution

Theorem (HCM'03,'07)

Subject to technical conditions, the MFG system has a unique solution for which the resulting set of MFG controls

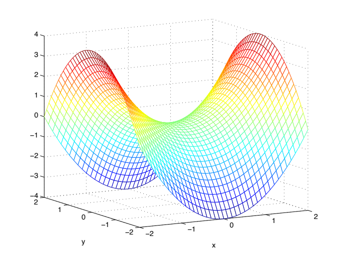
$$\mathcal{U}_{mf}^N = \{u_i^0 = -\frac{b}{r}(\Pi_i z_i + s_i); 1 \leq i \leq N\}, \;\; 1 \leq N < \infty$$

yields an *ϵ-Nash equilibrium for all ϵ*, *i.e.* $\forall \epsilon > 0 \;\exists N(\epsilon)$ *s.t.* $\forall N \geq N(\epsilon)$

$$J_i(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i} J_i(u_i, u_{-i}^0) \leq J_i(u_i^0, u_{-i}^0)$$

where u_i is adapted to the set of full information admissible controls.

- Agent y is a maximizer
- Agent x is a minimizer



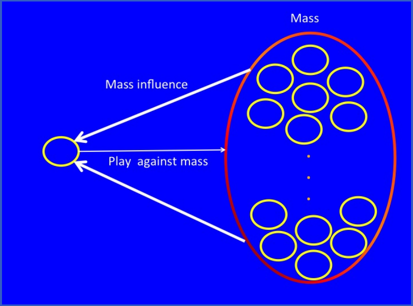
Background – Properties of MFG-LQG Solution

Counterintuitive Nature of MFG controls:

- Intrinsically decentralized agent's feedback $=$ feedback of agent's local stochastic state $+$ feedback of deterministic precomputable mass (No communication among agents!)

Applying MFG Controls to the Finite Population System:

- ϵ-Nash equilibrium (with respect to all possible controls among the full information pattern) exists between the individuals of a large N population system with $\epsilon \to 0$ as N goes to infinity



Background – Standard Consensus Algorithms

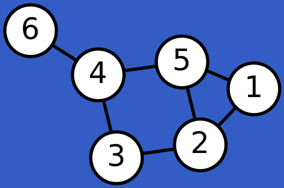
Definition

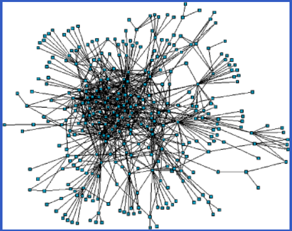
A consensus process is a process for achieving an agreement among the members of a group of agents on some common state property such as velocity or information.

Standard Consensus Algorithms (SCAs):
A network of N agents with dynamics

$$dz_i(t) = u_i(t)dt, \quad t \geq 0, \qquad 1 \leq i \leq N,$$

where an agreement is achieved via local communications with their neighbours based on the network topology $G = (V, E)$ (V: the set of vertices, $E \subset V \times V$: an ordered set of edges)





Background – Standard Consensus Algorithms

Time-Invariant SCAs:

$$dz_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}\big(z_j(t) - z_i(t)\big)dt, \quad t \geq 0, \qquad 1 \leq i \leq N$$

where $\mathcal{N}_i = \{j \in V : (i,j) \in E\}$.

Definition

Consensus is said to be achieved asymptotically for a group of N agents if $\lim_{t\to\infty} |z_i(t) - z_j(t)| = 0$ for any i and j, $1 \leq i \neq j \leq N$.

Theorem (see e.g. (Ren et.al. '05))

If the undirected graph G is connected (i.e., there is a path between every pair of nodes), then

- *the system achieves consensus asymptotically as time goes to infinity*
- *the consensus value is the average of initial states $\frac{1}{N}\sum_{j=1}^{N} z_j(0)$.*

Why MFG Consensus Formulation?

- The connectivity of the network structure needed for the SCAs (even for the less demanding "frequently connected" hypotheses) may not hold
- Communication is costly and may be distorted
- SCAs are fragile in the presence of noise in the agents' dynamics
- MFG approach with no communication but prior statistical information
- In this approach we seek to synthesize the collective behaviour of the group from fundamental principles

MFG Consensus Formulation – Homogenous Case

- Dynamics:

$$dz_i(t) = u_i(t)dt + \sigma dw_i(t), \quad t \geq 0, \qquad 1 \leq i \leq N$$

- Cost Functions:

$$J_i^N(u_i, u_{-i}) := E \int_0^\infty e^{-\rho t} \Big(\big(z_i(t) - \frac{1}{N-1} \sum_{j=1, j \neq i}^{N} z_j(t) \big)^2 + r u_i^2(t) \Big) dt$$

N; population size, z_i: state of agent i, u_i: control input
w_i: disturbance (standard Wiener process), $\rho > 0$: discount factor
$r > 0$: control penalty

Each agent in the group seeks a strategy to be as close as possible to the average of the population

- Let $F(\cdot)$ be the limit empirical distribution of $\{z_i(0) : i > 1\} \subset \mathcal{C}$

MFG Consensus Solution – Homogenous Case

Mean Field Game System of the Consensus Formulation:

- Computation of Best Response Control for a Generic Agent with Initial $\alpha \in \mathcal{C}$ and Mass Trajectory $\phi^\infty(\cdot)$:

$$u_\alpha^o(t) = -\frac{1}{r}\big(pz_\alpha(t) + s(t)\big) \qquad \text{(Best Response Control)}$$

$$p^2 + r\rho p - r = 0 \Longrightarrow p = (-r\rho + \sqrt{(r\rho)^2 + 4r})/2 \qquad \text{(Riccati Equation)}$$

$$\frac{ds(t)}{dt} = (\rho + \frac{p}{r})s(t) + \phi^\infty(t) \qquad \text{(Tracking equation)}$$

- Mass behavior equation in the consensus formulation under $u_\alpha^o(\cdot)$:

$$dz_\alpha(t) = -\frac{1}{r}\big(pz_\alpha(t) + s(t)\big)dt + \sigma dw_i(t) \qquad \text{(The generic agenet process)}$$

$$\frac{d\bar{z}_\alpha(t)}{dt} = -\frac{1}{r}\big(p\bar{z}_\alpha(t) + s(t)\big),\ \bar{z}_\alpha(0) = \alpha \qquad \text{(The mean state equation)}$$

$$\phi^\infty(t) = \int_{\mathcal{C}} \bar{z}_\alpha(t)dF(\alpha),\ t \geq 0 \qquad \text{(The mass function)}$$

MFG Consensus Solution – Homogenous Case

Theorem (NCMH'10)

The unique solution of MFG system: $(s(t), \phi^\infty(t)) = (-p\phi^\infty(0), \phi^\infty(0))$, $t \geq 0$.

Applying the MFG control $u_i^o(t) = -\frac{p}{r}\big(z_i(t) - \phi^\infty(0)\big)$ yields:

$$z_i^o(t) = \phi^\infty(0) + e^{-\frac{p}{r}t}\big(z_i(0) - \phi^\infty(0)\big) + \sigma \int_0^t e^{-\frac{p}{r}(t-\tau)} dw_i(\tau), \; t \geq 0.$$

Definition

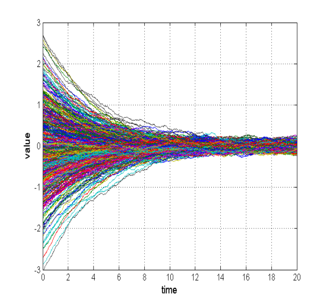
Mean-consensus is said to be achieved asymptotically for a group of N agents if $\lim_{t\to\infty} |\bar{z}_i(t) - \bar{z}_j(t)| = 0$ for any i and j, $1 \leq i \neq j \leq N$.

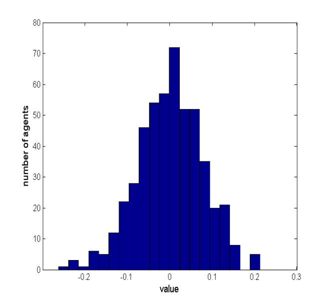
Theorem (NCMH'10)

(i) A mean-consensus is reached asymptotically as time goes to infinity with individual asymptotic variance $\frac{\sigma^2 r}{2p}$.
(ii) The set of MFG control strategies $\{u_i^o : 1 \leq i \leq N\}$ generates an ϵ_N-Nash equilibrium such that $\lim_{N\to\infty} \epsilon_N = 0$.

MFG Consensus Solution – Homogenous Case

Simulation Result (500 agents)





(A) Trajectories of agents' states, (B) Histogram of the system at time $t = 20$

MFG Consensus Formulation – Heterogeneous Case

- Dynamics:

$$dz_i(t) = u_i(t)dt + \sigma dw_i(t), \qquad 1 \le i \le N, \qquad l_i \in \Theta$$

l_i: type of agent i, $\Theta := \{\theta_1, \cdots, \theta_K\}$ to model K subpopulations,
N_k: the number of agents of type θ_k

- Cost Functions:

$$J_i^N(u_i, u_{-i}) := E\int_0^\infty e^{-\rho t}\Big(\big(z_i(t) - \frac{\sum_{j=1}^N \omega_{l_i l_j}^{(N)} z_j(t)}{\sum_{j=1}^N \omega_{l_i l_j}^{(N)}}\big)^2 + r u_i^2(t)\Big)dt,$$

with the weight coefficients:

$$\omega_{l_i l_j}^{(N)} = \begin{cases} 1/N_k & \text{for} \quad l_i, l_j = \theta_k, \\ \omega_{\theta_i \theta_j}/N_{k'} & \text{for} \quad l_i = \theta_k,\ l_j = \theta_{k'}. \end{cases}$$

where $\omega_{\theta_i\theta_j} \ge 0$ for any $\theta_i, \theta_j \in \Theta$ and $\sum_{j=1}^K \omega_{\theta_i\theta_j} \ne 0$ for each $\theta_i \in \Theta$.

MFG Consensus Solution – Heterogenous Case

Assumption: There exists a probability vector π such that

$$\lim_{N\to\infty}(\frac{N_1}{N},\cdots,\frac{N_k}{N}) = \pi := (\pi_1,\cdots,\pi_K)$$

where $\min_{1\le k\le K}\pi_k > 0$.

The Fundamental MFG System

$$\frac{ds_\theta(t)}{dt} = \big(\rho + \frac{p}{r}\big)s_\theta(t) + \phi_\theta^\infty(t), \quad \theta\in\Theta \qquad \text{(Tracking mass equation)}$$

$$\frac{d\bar{z}_\theta(t)}{dt} = -\frac{p}{r}\bar{z}_\theta(t) - \frac{1}{r}s_\theta(t), \quad \bar{z}_\theta(0) \qquad \text{(The mean state equation)}$$

$$\phi_\theta^\infty(\cdot) = \frac{\sum_{\theta'\in\Theta}\pi_{\theta'}\omega_{\theta\theta'}\bar{z}_{\theta'}(\cdot)}{\sum_{\theta'\in\Theta}\pi_{\theta'}\omega_{\theta\theta'}} \qquad \text{(The mass function)}$$

$$\text{Riccati Equation}: \quad p^2 + r\rho p - r = 0 \Longrightarrow p = (-r\rho + \sqrt{(r\rho)^2+4r})/2$$

- Individual best response action $u_\theta^o(t) = -\frac{1}{r}\big(pz_\theta(t) + s(t)\big)$ is optimal w.r.t tracked ϕ_θ^∞

MFG Consensus Solution – Heterogenous Case

Let

$$(W)_{ij} := \frac{\pi_j \omega_{\theta_i \theta_j}}{\sum_{k=1}^{K} \pi_k \omega_{\theta_i \theta_k}}, \qquad 1 \leq i, j \leq K$$

Matrix W is a row-stochastic matrix since all its row sums are 1.

Definition

A stochastic matrix is irreducible if its corresponding digraph is strongly connected.

Theorem

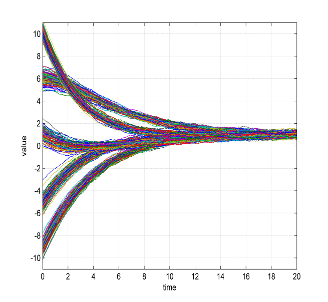
If W is irreducible then the unique stationary solution of the MFG system is

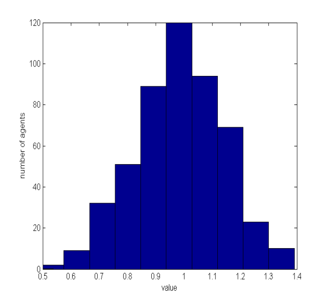
$$(s_\infty, \bar{z}_\infty) = \left(-p \frac{\gamma^T \bar{z}(0)}{\gamma^T \mathbf{1}_K} \mathbf{1}_K, \frac{\gamma^T \bar{z}(0)}{\gamma^T \mathbf{1}_K} \mathbf{1}_K \right)$$

where γ^T is the unique left-hand Perron vector for W. Hence, agents reach mean-consensus in $\frac{\gamma^T \bar{z}(0)}{\gamma^T \mathbf{1}_K} \mathbf{1}_K$.

MFG Consensus Solution – Heterogenous Case

Simulation Result: 500 agents in a system of 5 subpopulations such that W corresponds to an adjacency matrix of a strongly connected graph

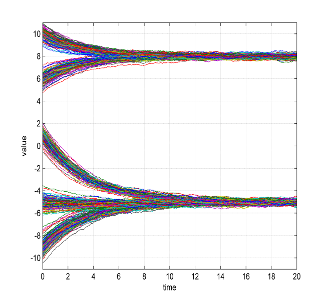


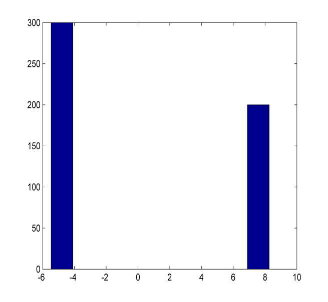


(A) Trajectories of agents' states, (B) Histogram of the system at time $t = 20$

MFG Consensus Solution – Heterogenous Case

Simulation Result: 500 agents in a system of 5 subpopulations such that W corresponds to an adjacency matrix of a graph with two connected components



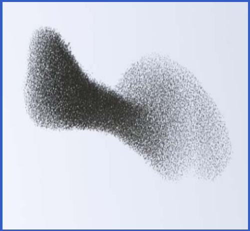


(A) Trajectories of agents' states, (B) Histogram of the system at time $t = 20$

Conclusion

Extensions and Generalizations:

- Analysis extends to the cooperative social optimization with the social cost $J_{\mathrm{soc}}^N(u) = \sum_{i=1}^N J_i^N(u_i, u_{-i})$
- Analysis extends to MFG flocking formulation



Future Research: Consensus algorithms by the use of:

- A priori statistical information (MFG)
- Local communications (SCAs)