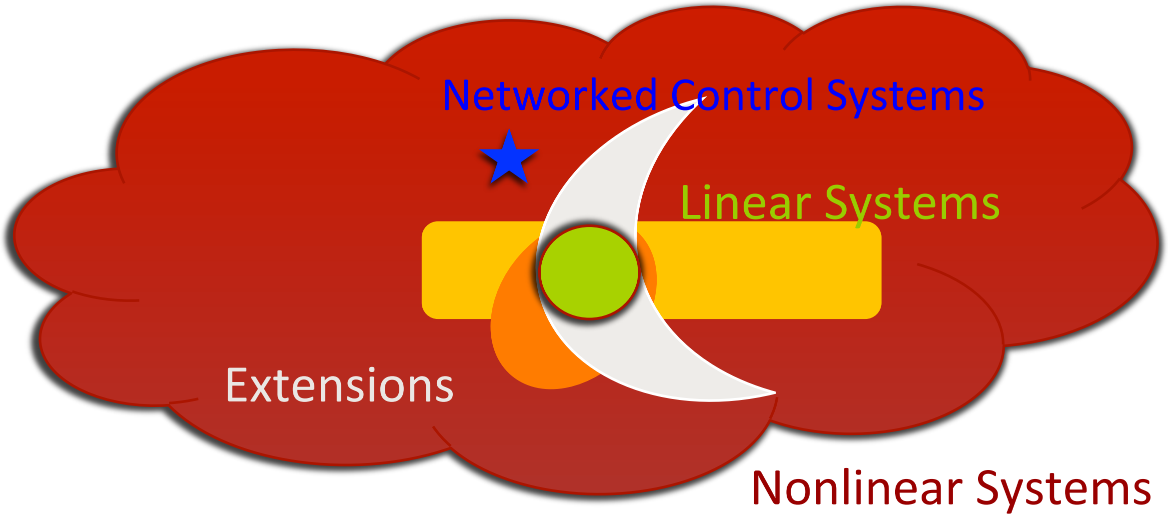
A New Framework for Stability Analysis of Networked Control Systems

Oct 16, 2012, Lund, Yumiko Ishido.

Research Interests: Analysis and Synthesis of Nonlinear Systems.



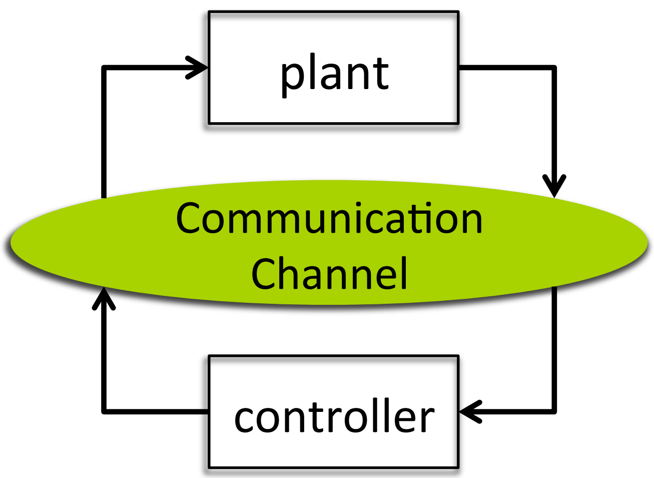
- Good tools for Linear Systems.
- Extensions for some classes of Nonlinear Systems.

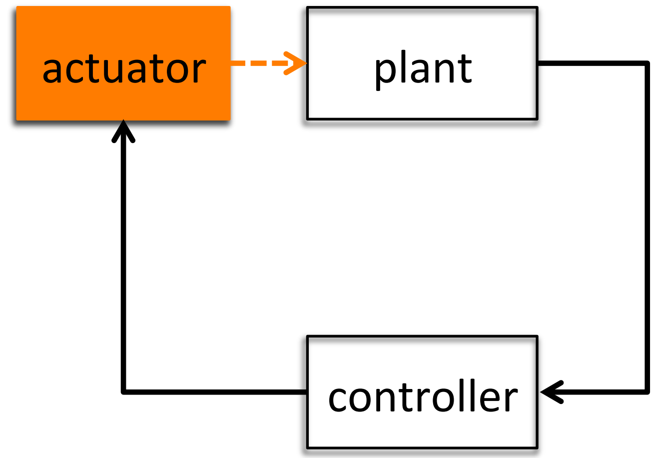
(Robust control based on small gain theorem, IQC approach, gain scheduling, etc…)

Goal: Develop a Mathematical Framework for Analysis and Synthesis of Networked Control Systems.

Networked Control Systems

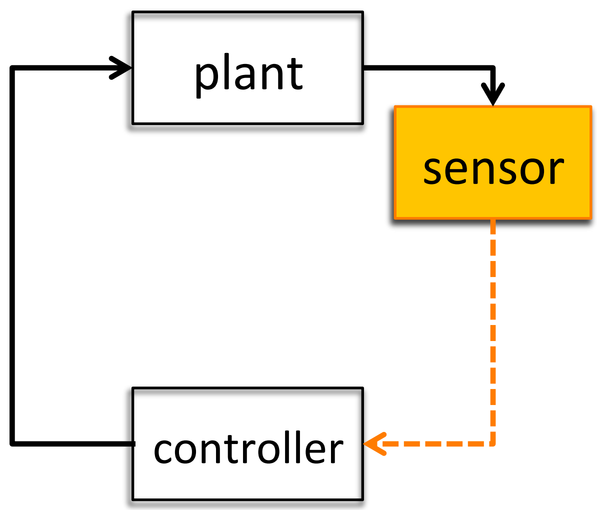
Involving a data rate-limited communication channel





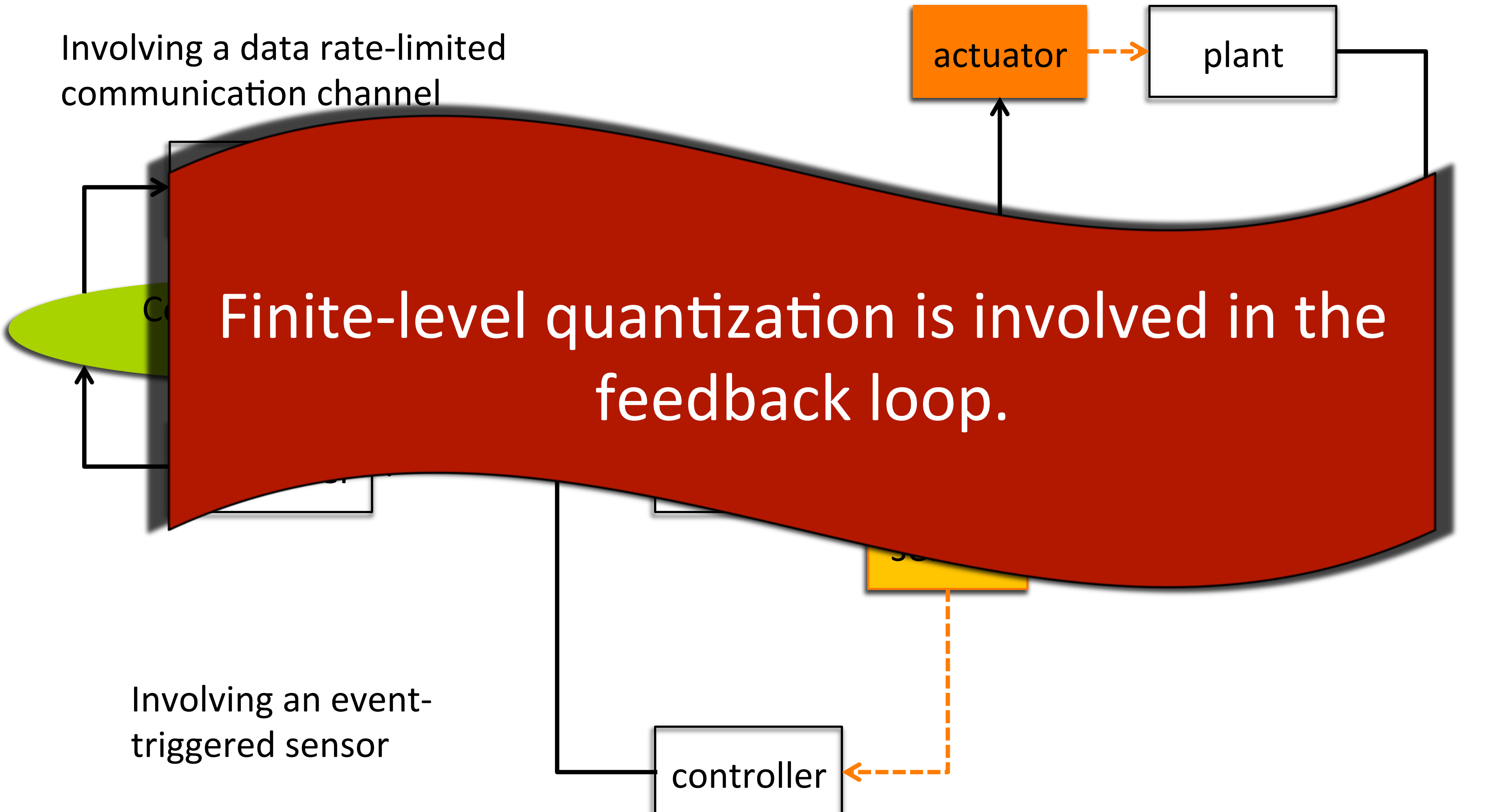
Involving a finite-level valued actuator

Involving an event-triggered sensor



Networked Control Systems

Involving a data rate-limited communication channel

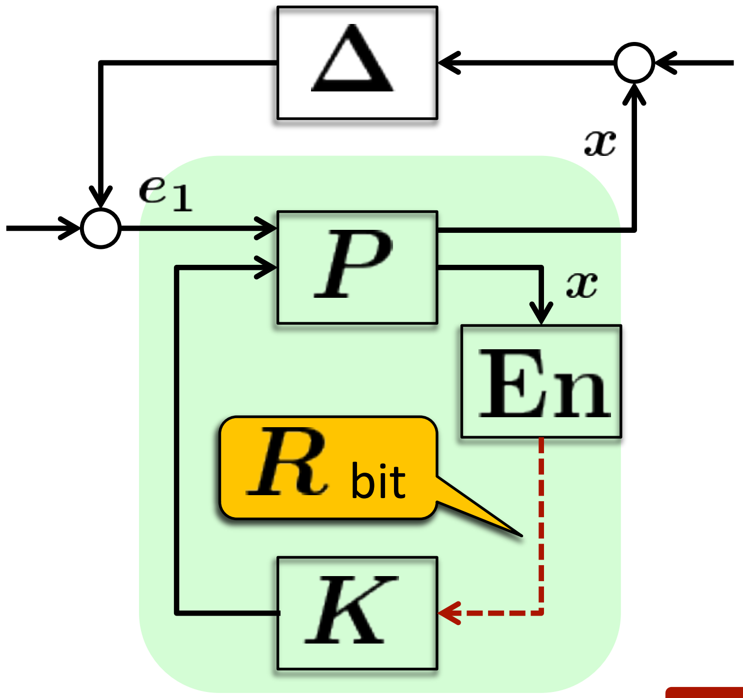


Finite-level quantization is involved in the feedback loop.

Involving an event-triggered sensor

Classical Framework does not work?

Ex1: Stabilization of an uncertain plant over a rate-limited communication channel.



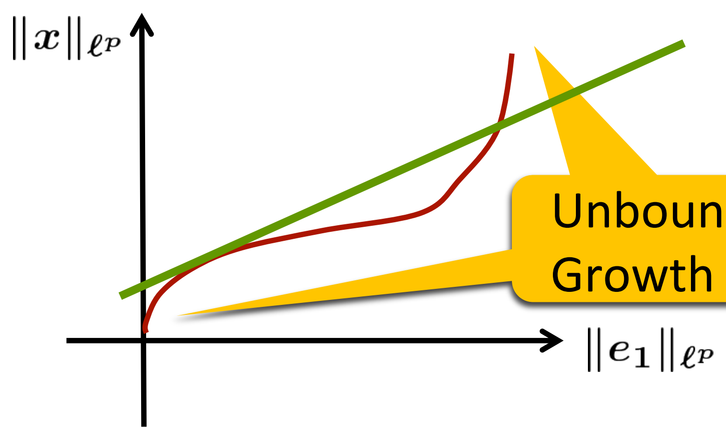
P : Unstable LTI

Δ : ℓ^p-gain bounded

Small Gain Theorem Is NOT Applicable!!

Achievable input-output property (Martins):

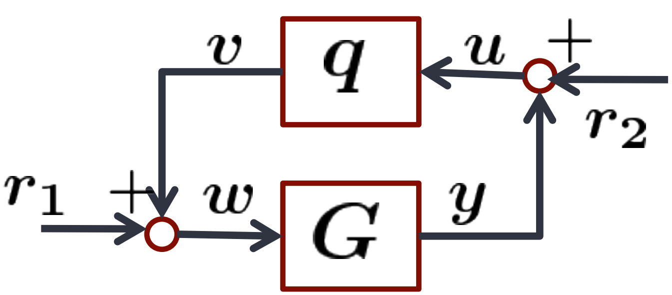
Suppose $\exists \alpha \in \mathcal{K}$ s.t. $\|x\|_{\ell^p} \leq \alpha(\|e_1\|_{\ell^p})$.

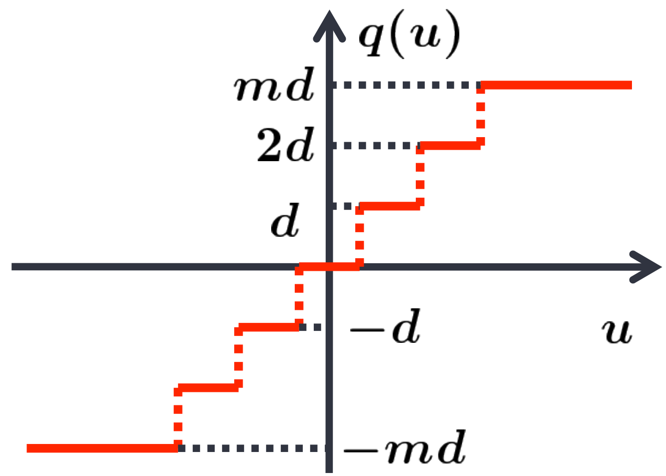


Need for introducing a practical Local Stability Analysis Framework.

Classical Framework does not work?

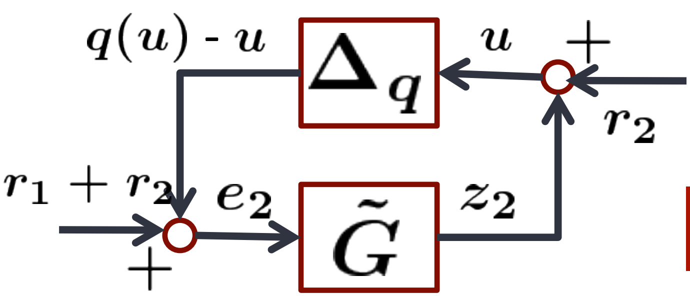
Ex2: Stability analysis of a feedback system involving a uniform quantizer.





G : SISO, LTI

q : uniform quantizer

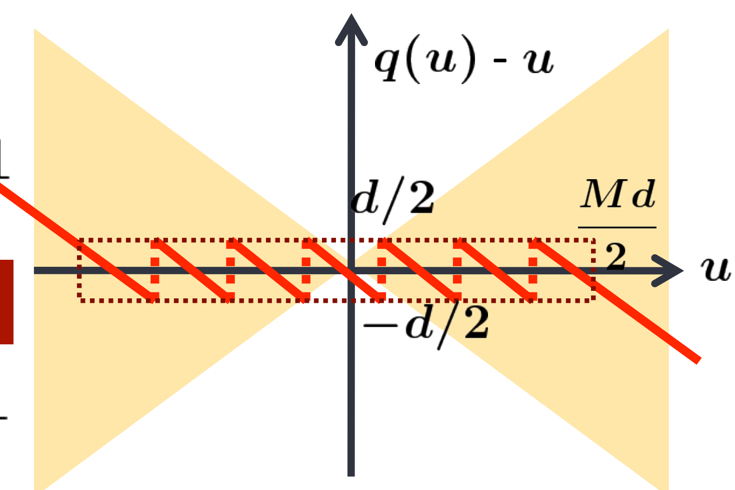


Δ_q : quantization error

$$\|\Delta_q\|_{\ell^\infty\text{-ind}} \leq 1$$

Stability Condition

$$\|\tilde{G}\|_{\ell^\infty\text{-ind}} < 1$$



A New Analysis Framework for Networked Control Systems

1. Introduce a reasonable notion of local stability for networked control systems.
2. Derive a key theorem for stability analysis.
3. Prepare a new class of nonlinearity that is suitable for expressing quantization errors.

Small ℓ^p Signal ℓ^p Stability

Small ℓ^p signal ℓ^p stability

Local Boundedness

A map H is said to be small ℓ^p signal ℓ^p stable with level γ and input bound ϵ if

$$\|u|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \gamma\epsilon$$

holds for given constants $\epsilon,\ \gamma > 0$.

Comparison with existing stabilities

ℓ^p stability $\exists \alpha \in \mathcal{K}, \beta \in \mathbf{R}_+$ such that

$$\|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \alpha(\|u|_{[0,\tau]}\|_{\ell^p}) + \beta$$

Finite gain ℓ^p stability $\exists \gamma, \beta \in \mathbf{R}_+$ such that

$$\|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \gamma\|u|_{[0,\tau]}\|_{\ell^p} + \beta$$

Small ℓ^p signal ℓ^p stability is...

- weaker than ℓ^p stability or finite gain ℓ^p stability
- equivalent when H is a linear map

Comparison with existing stabilities

Local ℓ^p stability (Bourles 1996) $\exists \epsilon, \gamma \in \mathbf{R}_+$ such that

$$\| u|_{[0,\tau]} \|_{\ell^p} \leq \epsilon \Rightarrow \| H(u)|_{[0,\tau]} \|_{\ell^p} \leq \gamma \| u|_{[0,\tau]} \|_{\ell^p}$$

Small signal ℓ^p stability (Vidyasagar & Vanelli, 1982)

$\exists c, \gamma \in \mathbf{R}_+$ such that

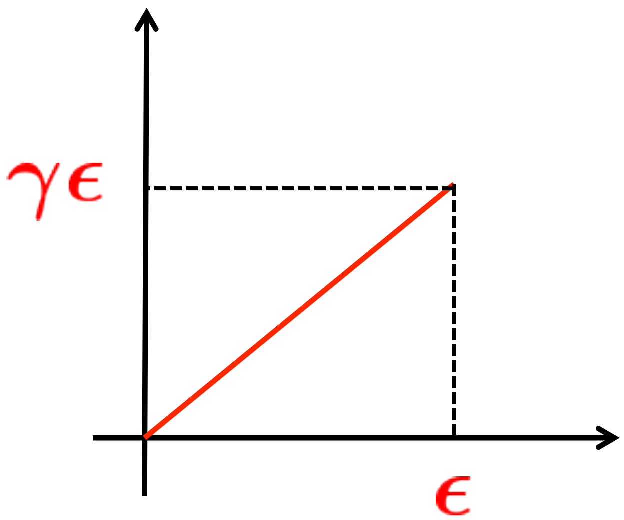
$$u \in \ell^p_e \cap \{ u \mid \|u\|_{\ell^\infty} \leq c \} \Rightarrow \| H(u)|_{[0,\tau]} \|_{\ell^p} \leq \gamma \| u|_{[0,\tau]} \|_{\ell^p}$$

Small ℓ^p signal ℓ^p stability is...

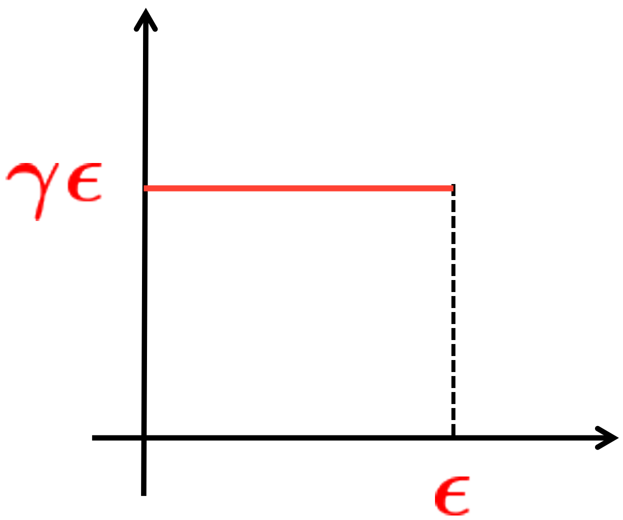
- Defined with local upper bounds on input-output signals (not defined with gain).

Comparison with existing stabilities

Local ℓ^p stability **vs** Small ℓ^p signal ℓ^p stability



“Local finite gain stability”

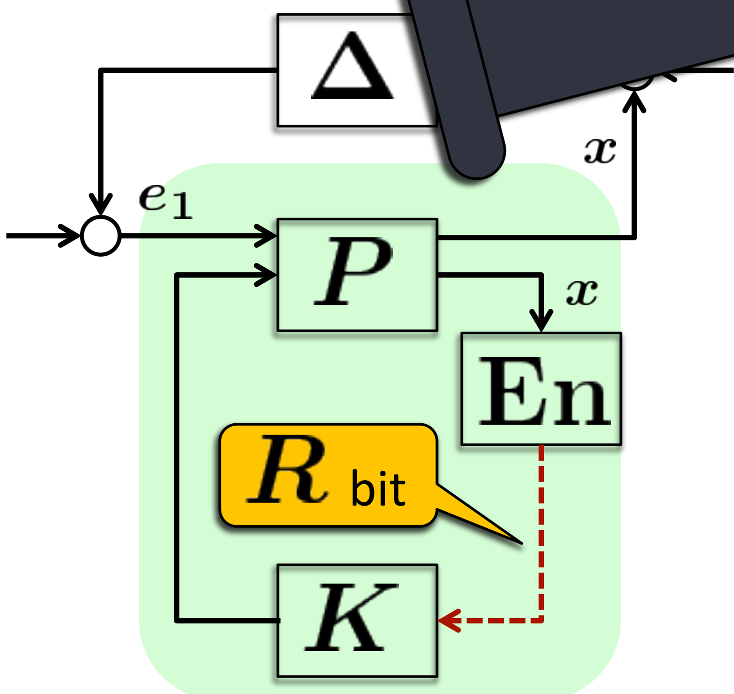


“Local boundedness”

Classical F… …work?

Recall!

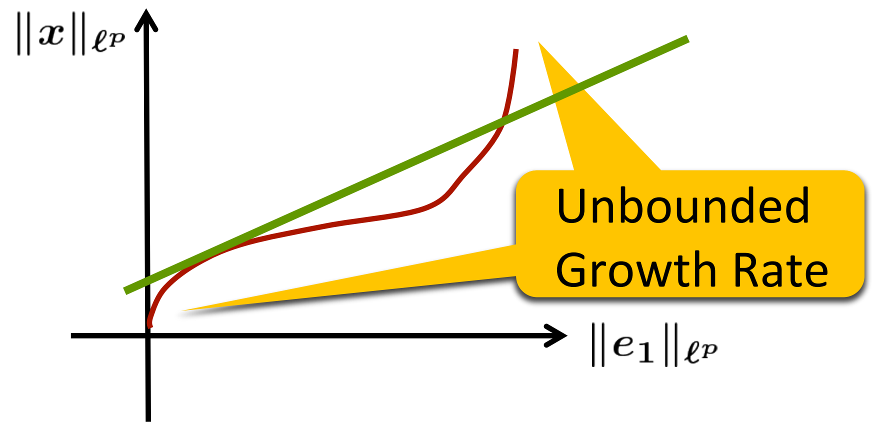
Ex1: Stabilizatio… …ted communication channel.



Small Gain Theorem In NOT Applicable!!

Achievable input-output property (Martins):

Suppose $\exists \alpha \in \mathcal{K}$ s.t. $\|x\|_{\ell^p} \leq \alpha(\|e_1\|_{\ell^p})$.



P : Unstable LTI

Δ : ℓ^p-gain bounded

Need for introducing a practical Local Stability Analysis Framework.

Comparison with existing stabilities

Input-to-output practical stability (Jiang et.al 1994)

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

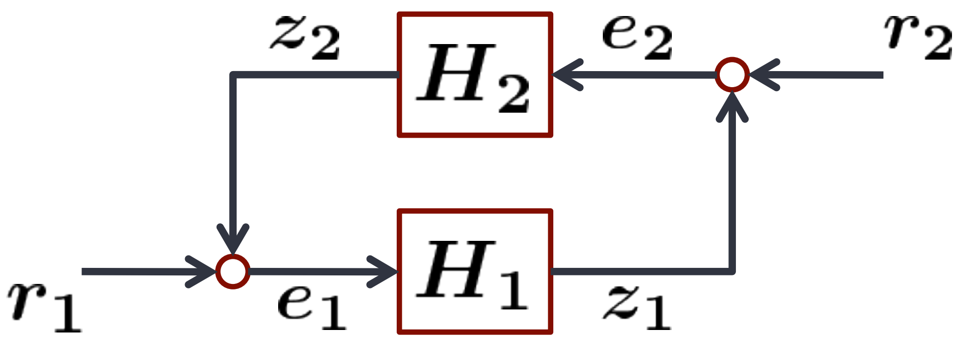
$\exists \beta \in \mathcal{KL},\ \gamma \in \mathcal{K},\ d \in \mathbf{R}_+$ such that

$$\|y(\tau)\|_\infty \leq \beta(\|x(0)\|_\infty, \tau) + \gamma(\|u|_{[0,\tau]}\|_{\mathcal{L}^\infty}) + d$$

Small ℓ^p signal ℓ^p stability is…

- Local stability notion.

Small ℓ^p Signal ℓ^p Stability



The feedback system is said to be small ℓ^p signal ℓ^p stable if there exist $\epsilon, \ \gamma > 0$ such that

$$\left\| \left. \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right|_{[0,\tau]} \right\|_{\ell^p} \leq \epsilon \Rightarrow \left\| \left. \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right|_{[0,\tau]} \right\|_{\ell^p} \leq \gamma\epsilon.$$

(Discrete time) Small Level Theorem

Small Level Theorem Assume the following conditions hold.

(i) H_1 : strictly causal & $\exists\ \epsilon_1,\ \gamma_1 > 0$ such that

$$\|e_1\|_{\ell^p} \leq \epsilon_1 \Rightarrow \|z_1\|_{\ell^p} \leq \gamma_1 \epsilon_1.$$

(ii) H_2 : $\exists\ \epsilon_2,\ \gamma_2 > 0$ such that

$$\|e_2\|_{\ell^p} \leq \epsilon_2 \Rightarrow \|z_2\|_{\ell^p} \leq \gamma_2 \epsilon_2.$$

(iii) $\gamma_1 \epsilon_1 < \epsilon_2$

(iv) $\gamma_2 \epsilon_2 < \epsilon_1$

⇨ $\gamma_1 \gamma_2 < 1$

Then the feedback system is small ℓ^p signal ℓ^p stable.

(Discrete time) Small Level Theorem

Small Level Theorem (Continued) In particular,

$$\left\| \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \Big|_{[0,\tau]} \right\|_{\ell^p} \leq \epsilon \Rightarrow$$

$$\left(\| z_1|_{[0,\tau]} \|_{\ell^p} \leq \delta_1 \text{ and } \| z_2|_{[0,\tau]} \|_{\ell^p} \leq \delta_2 \right)$$

$$\forall (r_1, r_2) \in \ell_e, \ \forall \tau \in \mathbb{Z}_+$$

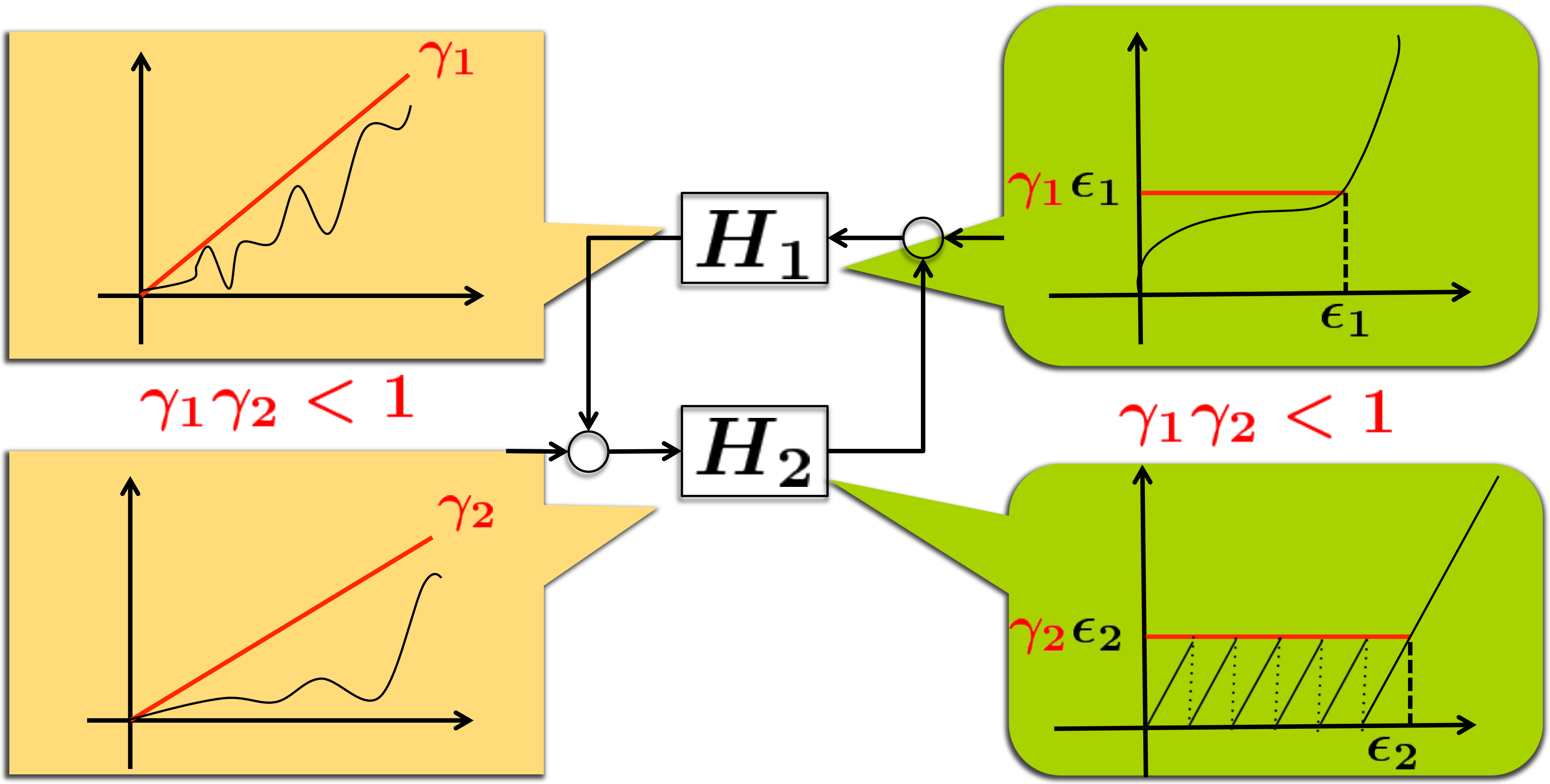
$$\epsilon := \min \left\{ \epsilon_2 - \gamma_1 \epsilon_1, \epsilon_1 - \gamma_2 \epsilon_2 \right\},$$

$$\delta_1 := \gamma_1 \epsilon_1, \quad \delta_2 := \gamma_2 \epsilon_2.$$

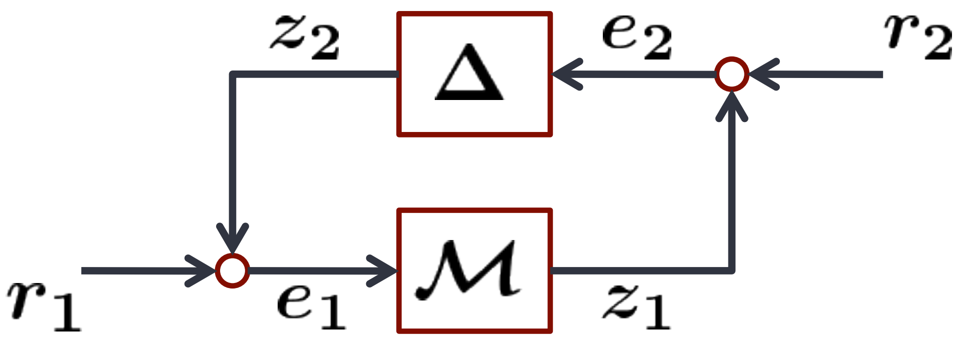
Small Gain Theorem vs Small Level Theorem

Small Gain Theorem

Small Level Theorem



Level Bounded Nonlinearity



Level bounded nonlinearity

Suitable for approximating quantization errors

$$\mathrm{SB}_{\Delta}^{\epsilon,\gamma} := \{\Delta \mid \|e_2|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \|\Delta(e_2)|_{[0,\tau]}\|_{\ell^p} \le \frac{\epsilon}{\gamma}\}$$

Theorem

Assume there exist $\epsilon_1, \gamma_1 < \gamma$ satisfying

(i) $\|e_1\|_{\ell^p} \le \epsilon_1 \Rightarrow \|z_1\|_{\ell^p} \le \gamma_1\epsilon_1 \quad \forall e_1 \in \ell^p$

(ii) $\frac{\epsilon}{\gamma} < \epsilon_1 < \frac{\epsilon}{\gamma_1}$

Then, the feedback system is small ℓ^p signal ℓ^p stable $\forall \Delta \in \mathrm{SB}_{\Delta}^{\epsilon,\gamma}$

A New Local Analysis Framework

Small ℓ^p signal ℓ^p stability

Local Boundedness

$$\|u|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \gamma\epsilon$$

γ : Attenuation level ϵ : Input bound

Small Level Theorem

If both subsystems have sufficiently **small level,** then the feedback system is small ℓ^p signal ℓ^p stable.

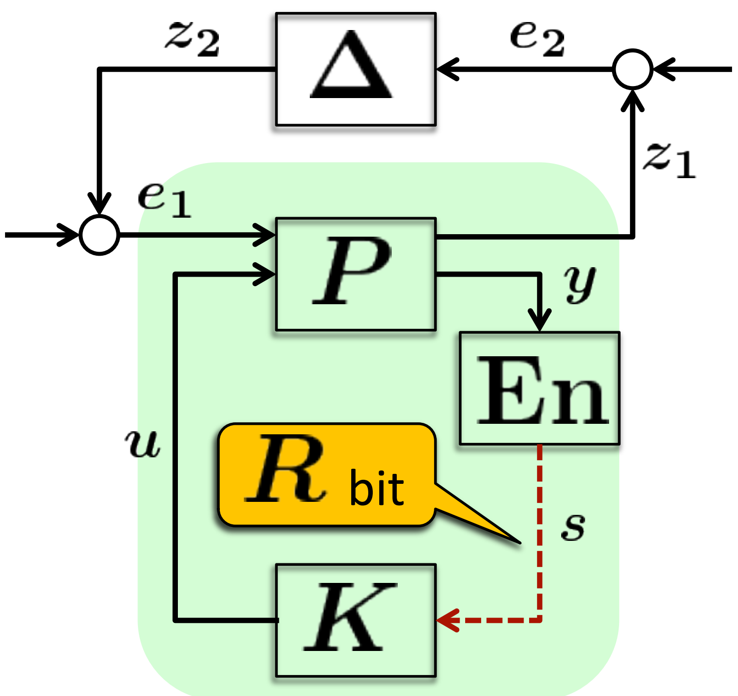
Level Bounded Uncertainty

Suitable for approximating quantization errors

$$\mathrm{SB}_{\Delta}^{\epsilon,\gamma} := \{\Delta \mid \|e_2|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|\Delta(e_2)|_{[0,\tau]}\|_{\ell^p} \leq \frac{\epsilon}{\gamma}\}$$

Quantitative Local Analysis Framework based on Local Boundedness.

Application Example 1



P : Unstable LTI

Δ : ℓ^p-gain bounded

Uncertain Plant

Nominal Plant (Unstable LTI): P

Uncertainty Δ : ℓ^p gain bouded

$$B_\Delta^\gamma := \{\Delta |$$

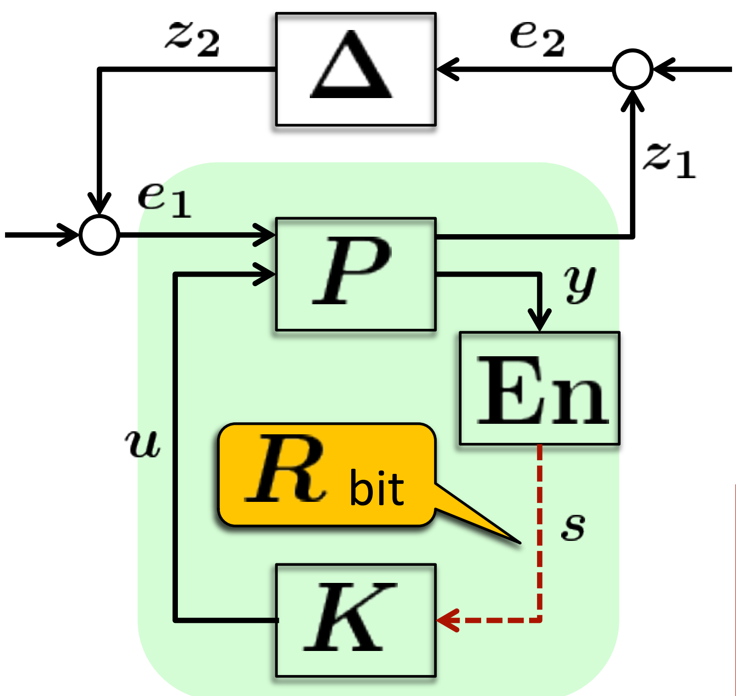
$$\|\Delta(e_2)|_{[0,\tau]}\|_{\ell^p} \leq \frac{1}{\gamma}\|e_2|_{[0,\tau]}\|_{\ell^p}, \forall\tau\}$$

Channel

$$s(t) = \{1, 2, \cdots, 2^R\}$$

En & K Causal maps

Application Example 1



P : Unstable LTI

Δ : ℓ^p-gain bounded

Small Level Condition If

$$\|e_1\|_{\ell^p} \leq \epsilon_1 \Rightarrow \|z_1\|_{\ell^p} \leq \hat{\gamma}\epsilon_1$$

holds for positive constants $\epsilon_1, \hat{\gamma} < \gamma$, the feedback system is small ℓ^p signal ℓ^p stable $\forall \Delta \in B_{\Delta}^{\gamma}$.

Sufficient condition on data rate R for the existence of (En, K) s.t. the small level condition hold.

(Necessary and sufficient condition for scalar nominal plant)

Application Example 1

Scalar Nominal Plant

$$x(t+1) = ax(t) + u(t) + e_1(t)$$
$$z_1(t) = cx(t)$$
$$y(t) = x(t)$$

Theorem Assume $\exists(\mathrm{En}, K)$ s.t. small level condition holds for some $\epsilon_1 > 0, \hat{\gamma} \in (0, \gamma)$, then R satisfies

$$|a| < 2^R, \quad \frac{|c|}{1 - |a|/2^R} \leq \gamma$$

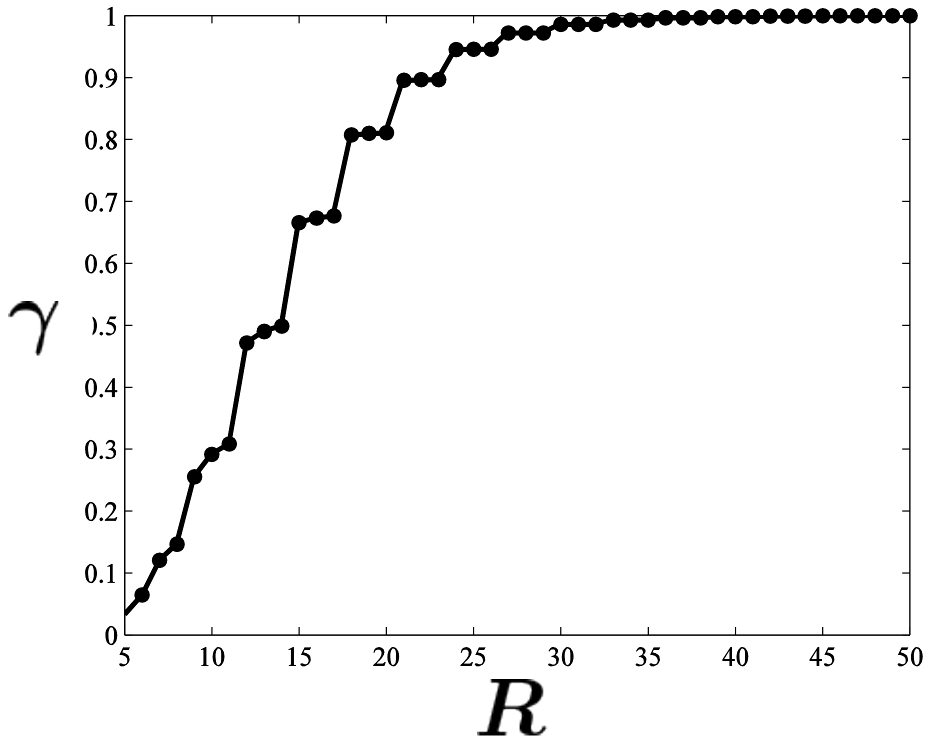
Conversely, if R satisfies

$$|a| < 2^R, \quad \frac{|c|}{1 - |a|/2^R} < \gamma$$

For any $\epsilon_1 > 0$, there exist (En, K) s.t. nominal part satisfies the small level condition.

Application Example 1

Admissible uncertainty



Data rate at the channel

Nominal Plant

$$x(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e_1(t), \; x(0) = 0$$

$$z_1(t) = \begin{bmatrix} 1 & \frac{17}{6} & \frac{19}{18}\sqrt{3} \end{bmatrix} x(t),$$

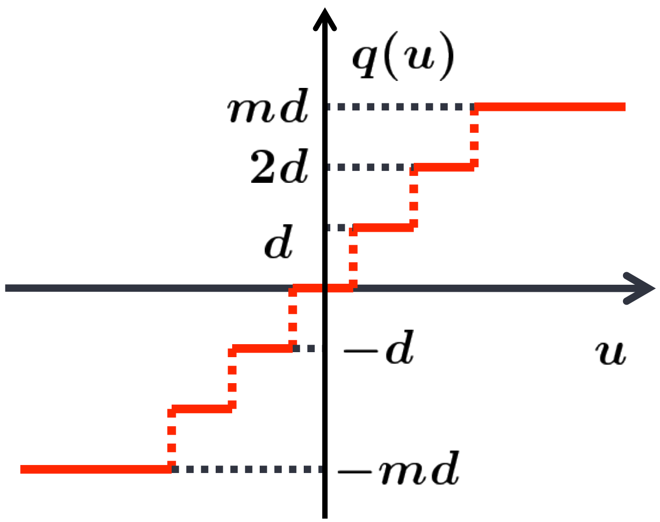
$$y(t) = x(t).$$

(Transfer Matrix)

$$P(z) = \frac{1}{(z-1)(z-2)(z+3)} \left[\begin{array}{c:c} \frac{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}{(z-1)(z+3)} & \frac{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}{(z-1)(z+3)} \\ \hdashline z & z \\ \sqrt{3} & \sqrt{3} \end{array} \right]$$

Trade-off between data rate and uncertainty

New Class of Nonlinearities



q : Uniform quantizer

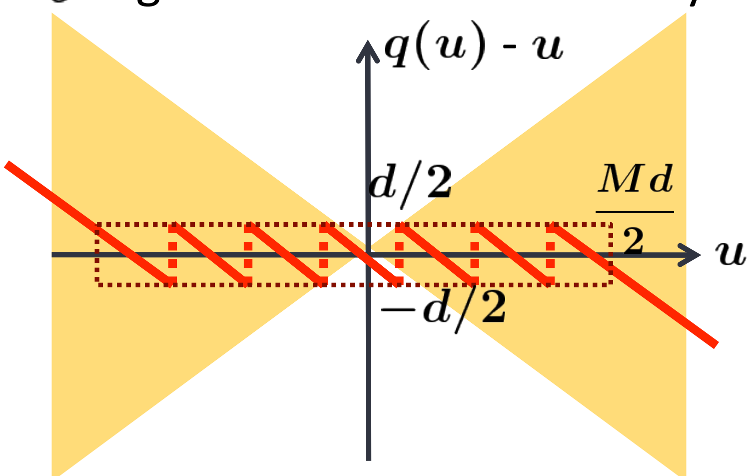
$\mathbb{R} \to V := \{0, \pm d, \cdots, \pm md\}$

Rounding input to the nearest output.

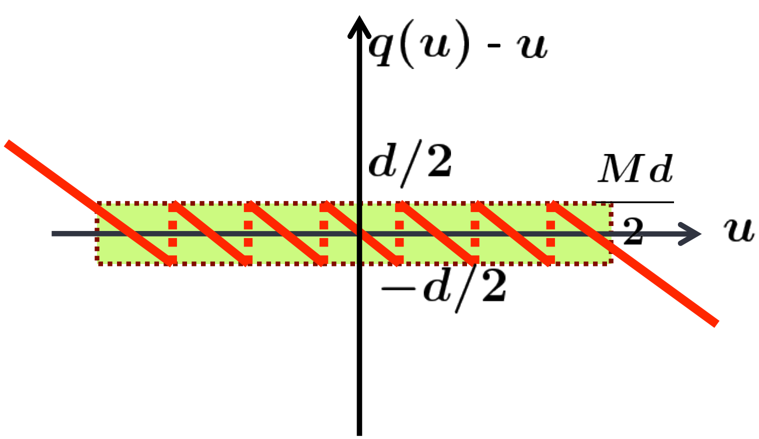
$d \in \mathbb{R}_+$: step size

$M := 2m + 1$: quantization levels

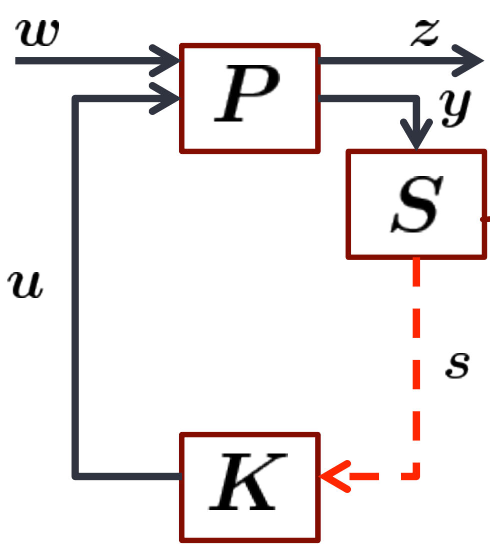
ℓ^∞ gain bounded nonlinearity



ℓ^∞ level bounded nonlinearity



Application Example 2



P, K : LTI systems

Event-triggered sensor

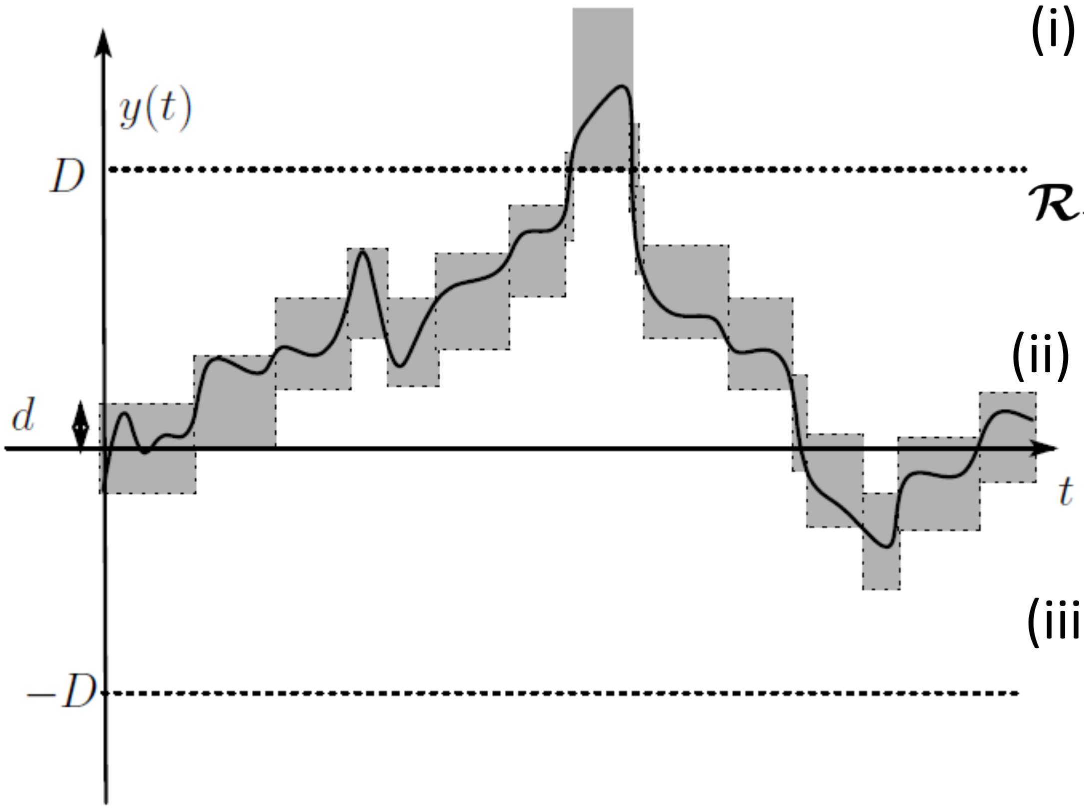
✓ Continuously observes y.

✓ Sends information to the controller only when y satisfies some condition.

Involves sampling rather than quantization.

Application Example 2

Scalar Nominal Plant $(\mathcal{R}_{track}(t), s(t))$ **Fixed-range triggered type**



(i) If $-D \le y(t) \le D$, then

$$s(t) := y(t),$$
$$\mathcal{R}_{track}(t) := [s(t) - d, s(t) + d].$$

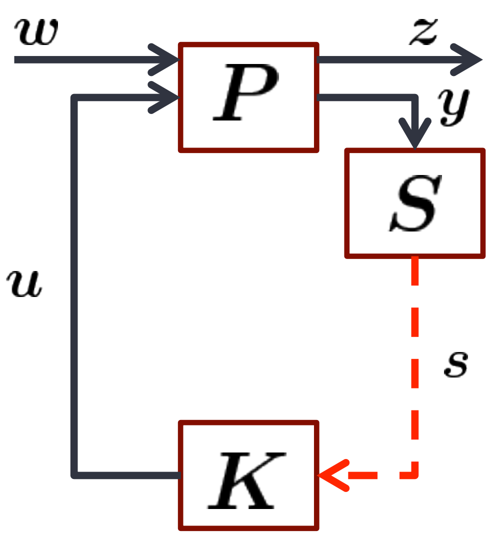
(ii) If $D < y(t)$, then

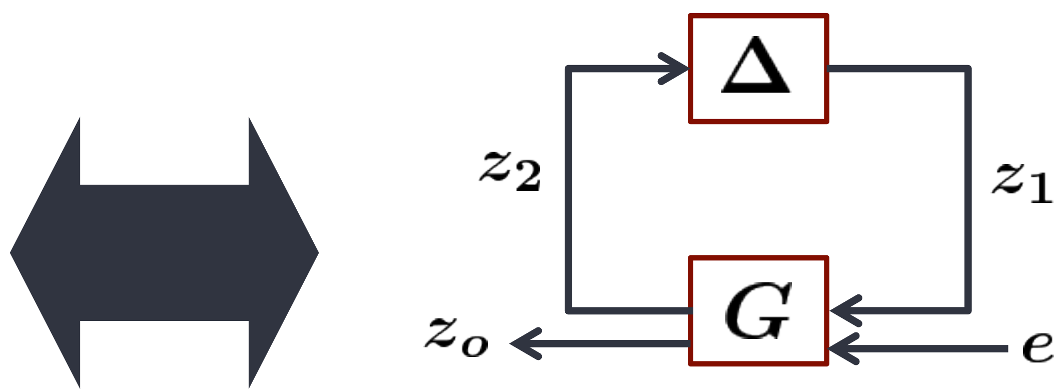
$$s(t) := D + d,$$
$$\mathcal{R}_{track}(t) := [D, \infty).$$

(iii) If $y(t) < -D$, then

$$s(t) := -D - d,$$
$$\mathcal{R}_{track}(t) := (-\infty, -D].$$

Application Example 2





Derive a condition on sensor parameters for local stability

Application Example 2

Theorem If

$$\frac{d}{D+2d}\gamma_{2z} < 1 \Leftrightarrow \gamma_{2z} - 2 < \frac{D}{d}$$

Then, the event-triggered system is small $\mathcal{L}^\infty$ signal $\mathcal{L}^\infty$ stable.
In particular,

$$\|w|_{[0,\tau]}\|_{\mathcal{L}^\infty} \le \epsilon := \frac{D + d(2-\gamma_{2z})}{\gamma_{2e}} \Rightarrow$$

$$\|z|_{[0,\tau]}\|_{\mathcal{L}^\infty} \le \gamma'_{2z} d + \gamma_{2e} \|w|_{[0,\tau]}\|_{\mathcal{L}^\infty}$$

Application Example 2

Numerical Example

Plant

$$P : \begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t), \\ y(t) = z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases}$$

Controller 1

$$K_1 : \begin{cases} \dot{x}_K(t) = \begin{bmatrix} -10 & 0 \\ 1 & 0 \end{bmatrix} x_K(t) + \begin{bmatrix} 16 \\ 0 \end{bmatrix} \tilde{y}(t), \\ u(t) = \begin{bmatrix} 18.71 & -0.375 \end{bmatrix} x(t) - 33\tilde{y}(t), \end{cases}$$

Stability Condition

Any positive d and D are OK.

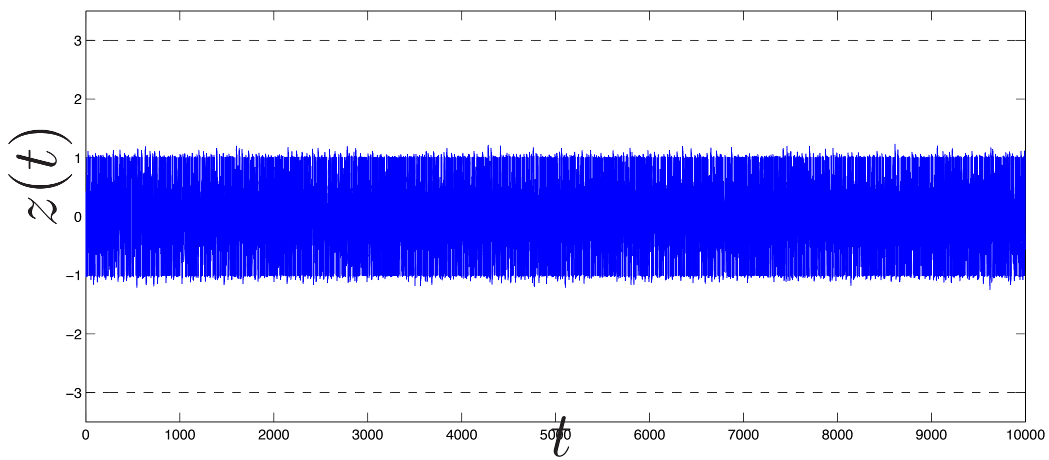
Norm bounds

$$D = d = 1 \quad ➡ \quad \|w|_{[0,\tau]}\|_{\mathcal{L}^\infty} \leq 2.7294 \Rightarrow \|z|_{[0,\tau]}\|_{\mathcal{L}^\infty} \leq 3$$

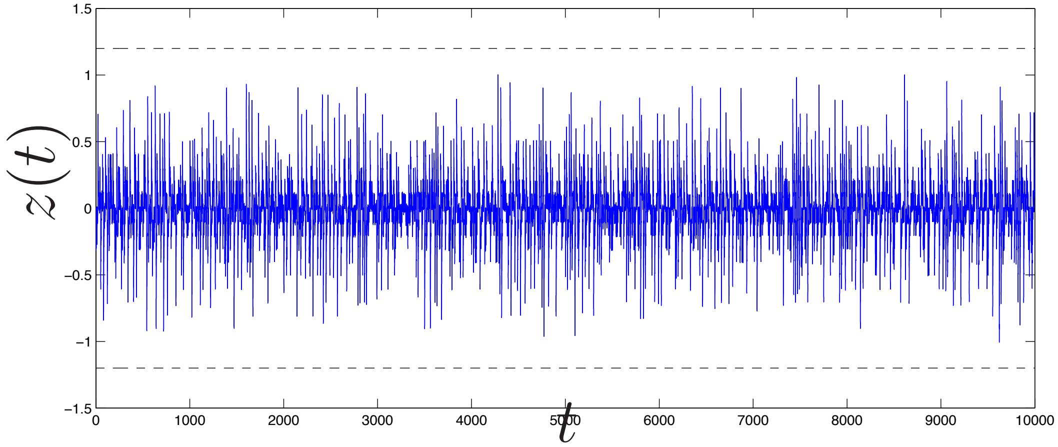
$$D = 1,\ d = 0.1 \quad ➡ \quad \|w|_{[0,\tau]}\|_{\mathcal{L}^\infty} \leq 1.9002 \Rightarrow \|z|_{[0,\tau]}\|_{\mathcal{L}^\infty} \leq 1.2$$

Application Example 2

$$D = d = 1 \;\; ➡$$



$$D = 1, \;\; d = 0.1 \;\; ➡$$



Application Example 2

<u>Numerical Example</u>

Plant

$$P : \begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t), \\ y(t) = z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases}$$

Controller 2

$$K_2 : \begin{cases} \dot{x}_K(t) = \begin{bmatrix} -10 & 0 \\ 1 & 0 \end{bmatrix} x_K(t) + \begin{bmatrix} 14 \\ 0 \end{bmatrix} \tilde{y}(t), \\ u(t) = \begin{bmatrix} 14 & -0.2 \end{bmatrix} x(t) - 55\tilde{y}(t), \end{cases}$$

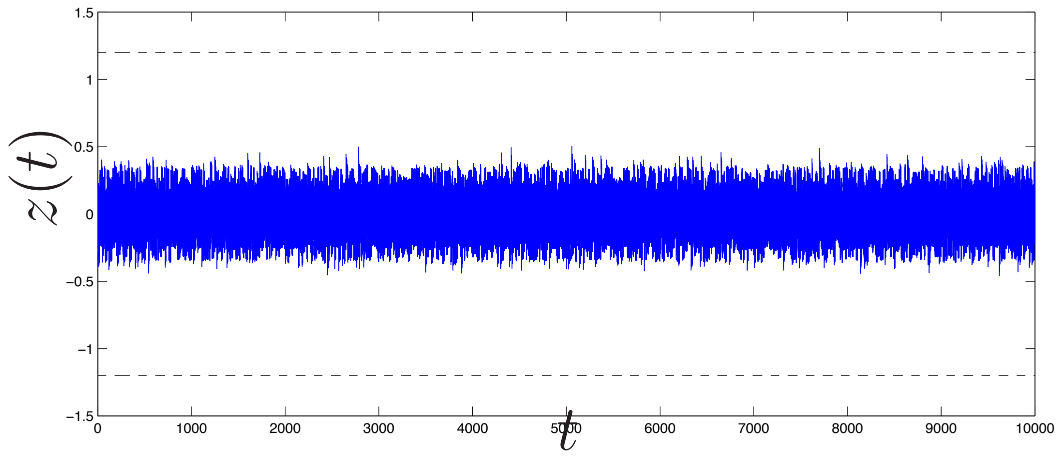
Stability Condition

$$3.9864 < \frac{D}{d}$$

Norm bounds

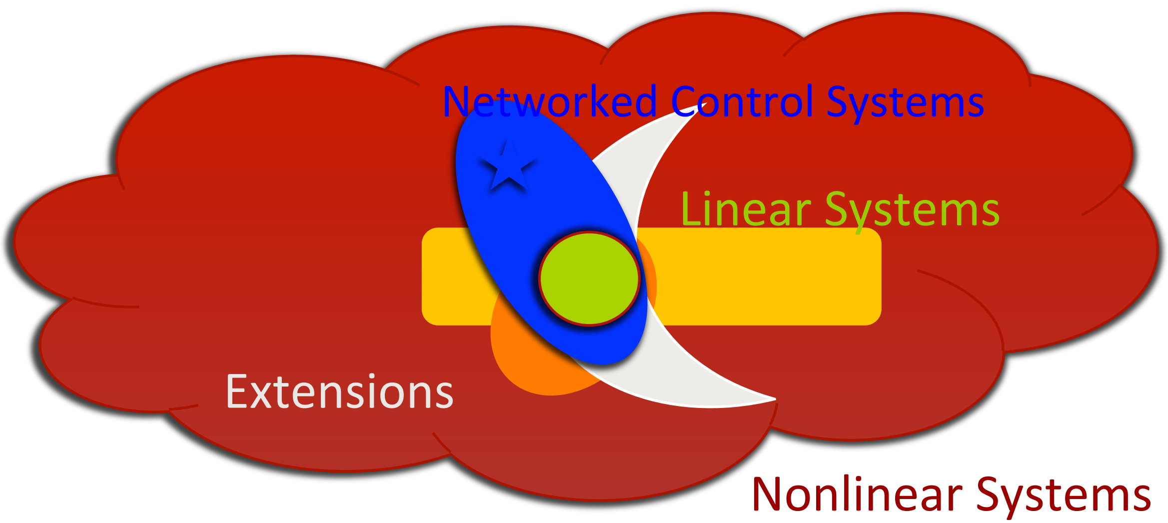
$$D = 1, \ d = 0.1 \Rightarrow \|w|_{[0,\tau]}\|_{\mathcal{L}\infty} \leq 3.5347 \Rightarrow \|z|_{[0,\tau]}\|_{\mathcal{L}\infty} \leq 1.2$$

Application Example 2



Conclusions

Research Interests: Analysis and Synthesis of Nonlinear Systems.



- Local analysis framework for networled control systems
- Extension to continuous-time hybrid systems

<u>Possible future work:</u>

1. Lyapunov approach: relation with internal stabilities. Focusing on a bounded band?
2. Analysis of stabilizable range for a locally stabilizing controller.