# Necessy and suffint conditions for stabilizablity in decentralized control and estimation

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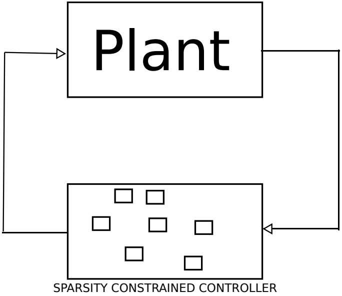
*LCCC workshop in Information and Control in Networks, Oct. 18, 2012*

# Theme 1




*classical*

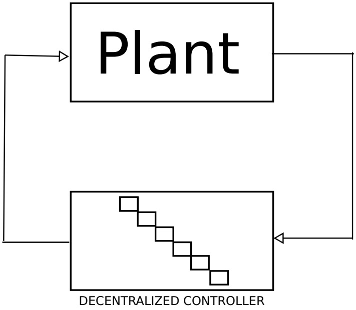
Factorization Approaches:

- Youla's parametrization,
- Model matching,
- Exact model matching

*new results*

- ✡ Necessary and sufficient conditions for stabilizability
- ✡ A new sparsity constrained version of Youla's parametrization

# Theme 2




*classical*

Characterization of stabilizability:

- Fixed Modes,
- Structured fixed modes,
- Quotient fixed modes

*new applications*

✡ Distributed State Estimation

1. Motivation

2. Theme 1
   - Norm optimal control

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   - Norm optimal control
     - Centralized

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     - Sparsity Constrained

1 Motivation

2 Theme 1
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- Open Questions

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   - A Meaningful, Particular Case
3. Theme 2

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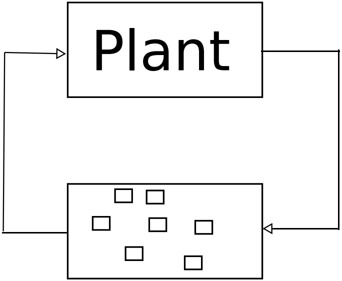
3. Theme 2
   - Problem formulation

# Theme 1

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☆ Necessary and sufficient conditions for stabilizability

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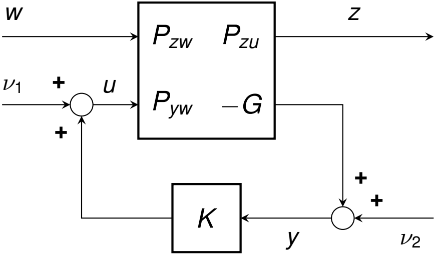
*S. Sabau and N. C. Martins, "Stabilizability and Norm-Optimal Control Design subject to Sparsity Constraints," arXiv:1209.1123*

# Problem Formulation: centralized



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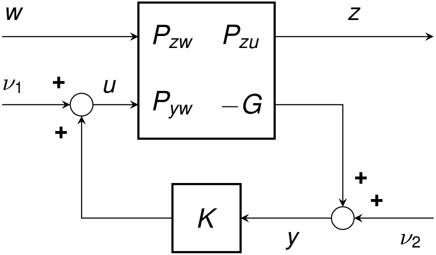


Here $P$ is a LTI plant, and $G$ is realized as: $G(\lambda) \stackrel{def}{=} C(\lambda I - A)^{-1}B + D$

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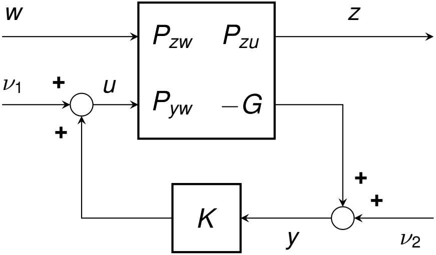


Here $P$ is a LTI plant, and $G$ is realized as: $G(\lambda) \overset{def}{=} C(\lambda I - A)^{-1}B + D$

## Problem

*Norm-optimal design is formulated as:*

$$\min_{K \text{stabilizes } P} \left\| P_{zw} + P_{zu}\, K\left(I + GK\right)^{-1} P_{yw} \right\|$$

# Convex Solutions

## Problem

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Solutions:

# Convex Solutions

## Problem

$$\min_{K \text{stabilizes } P} \left\| P_{zw} + P_{zu} K (I + GK)^{-1} P_{yw} \right\|$$

Solutions:

- Doubly-coprime factorization of $G$ (Nett, Jacobson and Ballas '84) $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

# Convex Solutions

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- Youla parametrization of stabilizing controllers

$$\begin{aligned} K &= -\left(\tilde{X} + MQ\right)\left(\tilde{Y} - NQ\right)^{-1} \\ &= \left(Y - Q\tilde{N}\right)^{-1}\left(X + Q\tilde{M}\right) \end{aligned}$$

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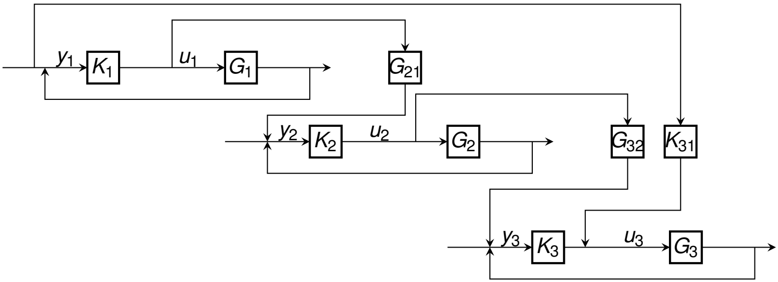
- Model matching formulation $\min_{Q \text{ stable}} \|T_1 + T_2 Q T_3\|$

# Sparsity Patterns: Pattern and Sparse Operators




$$G = \begin{bmatrix} [G]_1 & O & O \\ [G]_{21} & [G]_2 & O \\ O & [G]_{32} & [G]_3 \end{bmatrix}, \quad K = \begin{bmatrix} [K]_1 & O & O \\ O & [K]_2 & O \\ [K]_{31} & O & [K]_3 \end{bmatrix}$$

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## Definition

Denote $\text{Pattern}(G)$ with $G^{bin}$ and $\text{Pattern}(K)$ with $K^{bin}$.

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**Definition**

Denote $\text{Pattern}(G)$ with $G^{bin}$ and $\text{Pattern}(K)$ with $K^{bin}$.

**Definition**

Denote the set of stabilizing, sparsity constrained controllers (i.e. satisfying $K^{bin}$) with $\mathcal{S}$.

# Norm-optimal control: Sparsity constrained

## Problem

*Given $P$ and an appropriate pre-selected $K^{bin}$:*

$$\min_{K \in \mathcal{S}} \left\| P_{zw} + P_{zu} \, K \left( I + GK \right)^{-1} P_{yw} \right\|$$

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Main obstacles:

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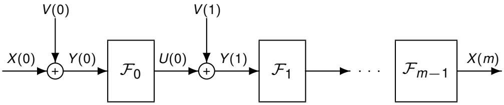
- Optimal controllers may be non-linear (Witsenhausen '68).
- Simple sequential linear quadratic Gaussian problems have non-linear optimal solutions. (Lipsa & Martins, Automatica '10)




$$\min E \left[ (X(m) - X(0))^2 + \varrho \sum_{i=0}^{m-1} U(i)^2 \right]$$

# Sample Key Results

- H. S. Witsenhausen, A counterexample in stochastic optimum control, SIAM Journal of Control, vol. 6, no. 1, pp. 131147, 1968.

**Main Idea:** Optimal solutions for certain lqg problems may be nonlinear

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**Main Idea:** Characterization of partially nested ...

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**Main Idea:** Fixed modes ...

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**Main Idea:** Cost also matters ...

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# A Sample of Existing Work and Recent Results


- A. B. Ozguler “Decentralized control: A stable proper fractional approach”, *IEEE Transactions on Automatic Control*, Vol. 35, No. 10, pp. 1109-1117, October 1990
- K. A. Unyelioglu and A. B. Ozguler “Decentralized stabilization: Characterization of all solutions and genericity aspects”, *International Journal of Control*, Vol. 55, No. 6, pp. 1381-1403, 1992
- K. A. Unyelioglu, A. B. Ozguler, “Decentralized blocking zeros and the decentralized strong stabilization problem”, *IEEE Transactions on Automatic Control*, vol. AC-40, No. 11, pp. 1905-1918, November 1995.
- R. A. Date and J. H. Chow, ”Decentralized stable factors and a parameterization of decentralized controllers,” *IEEE Trans. Automatic Control*, vol. 39, pp. 347351, 1994.

# A Sample of Existing Work and Recent Results

- P. G. Voulgaris. Control under structural constraints: An input-output approach. In *Lecture notes in control and information sciences*, pages 287–305, 1999
- X. Qi, M. Salapaka, P. Voulgaris and M. Khammash, *Structured Optimal and Robust Control with Multiple Criteria: A Convex Solution*, IEEE Trans. Aut. Control, Vol.49, No.10, pp 1623–1640, 2004
- B. Bamieh, P. G. Voulgaris, *A Convex Characterization of Distributed Control Problems in Spatially Invariant Systems with Communication Constraints*, Systems and Control Letters 54 (2005), pp. 575 – 583
- M. Rotkowitz and S. Lall, *A Characterization of Convex Problems in Decentralized Control*, IEEE Trans. Aut. Control, Vol.51, No.2, 2006. (pp. 274-286)"
- L. Lessard and S. Lall, *Quadratic invariance is necessary and sufficient for convexity,* American Control Conference, pp. 5360-5362, July 2011
- P. Shah and P. A. Parrilo, " An optimal controller architecture for poset-causal systems," IEEE CDC 2011

# Most related results and concepts

$$G = \begin{bmatrix} [G]_{11} & O & O \\ [G]_{21} & [G]_{22} & [G]_{23} \\ [G]_{31} & [G]_{32} & [G]_{33} \end{bmatrix}, \; K \in \begin{bmatrix} \star & O & O \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

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How do we check if the sparsity constraints allow for a convex parametrization?

Answer: If and only if the following holds (Rotkowitz & Lall):

$$KGK \in \mathcal{S} \quad \text{for all} \quad K \in \mathcal{S}$$

The following is a key invariance identity:

$$K \in \mathcal{S} \iff K(I + GK)^{-1} \in \mathcal{S}$$

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Summary of key advantages of Quadratic Invariance:

- There may be a convex Q-parametrization of all stabilizing controllers.
- Linear controllers are optimal for norm-based formulations.
- It encompasses other characterizations that allow for convex parametrization of the sparsity constrained controllers.

# Open questions:

- Given a plant what is the sparsest constraint on the controller that preserves QI? State of the art: Rotkowitz & Martins, "On The Nearest Quadratically Invariant Information Constraint," IEEE Transactions On Automatic Control, Vol. 57, No. 5, May 2012, pp. 1314 - 1319.

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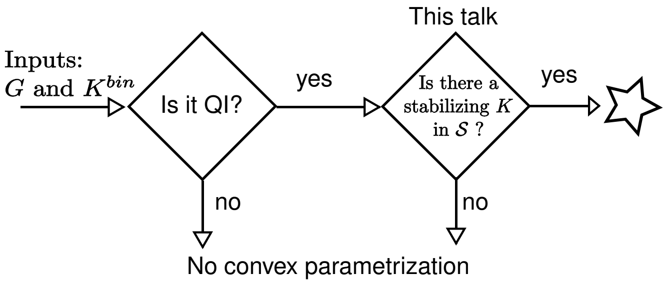
- (Main questions) Existing parametrizations of stabilizing controllers require an initial stabilizing controller. When does such a controller exist? When it does exist, how can it be computed? Can we characterize all sparsity-constrained stabilizing controllers in a way analogous to Youla's parametrization? (this talk)

# Necessary and Sufficient Conditions for Stabilizability under QI

## Problem

*Existing parametrizations of stabilizing controllers under QI rely on an initial stable stabilizing controller. When does a stabilizing controller exist? How do we compute it, if one does exist? Can we characterize all sparsity-constrained stabilizing controllers in a way analogous to Youla's parametrization?*



# Necessary and Sufficient Conditions for Stabilizability under QI

Doubly-Coprime Factorization of $G$ (Nett, Jacobson and Ballas '84)

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

**Theorem**

*Given G and a QI sparsity constraint $S$, there exists a stabilizing K in $S$ if and only if there exists some DCF of G such that*

$$\text{Pattern}(\tilde{X}\tilde{M}) \le K^{\text{bin}} \quad \text{or} \quad \text{Pattern}(MX) \le K^{\text{bin}}. \tag{1}$$

# Key idea behind the proof:

**Proposition**

*Given any DCF of G, select K to be the central controller* $K = \tilde{X}\tilde{Y}^{-1} = Y^{-1}X$. *The following identities hold:*

$$MX = K(I+GK)^{-1}, \qquad \tilde{X}\tilde{M} = K(I+GK)^{-1} \tag{2}$$

**Proof:** We verify that $MX = K(I+GK)^{-1}$ holds. The proof that $X\tilde{M} = K(I+GK)^{-1}$ is true is analogous. From $K = Y^{-1}X$ and $G = NM^{-1}$, we get that $K(I+GK)^{-1} = (I + Y^{-1}XNM^{-1})^{-1}Y^{-1}X$, where we used the fact that $K(I+GK)^{-1} = (I+KG)^{-1}K$. Finally, using Bézout's identity we get that $(I + Y^{-1}XNM^{-1}) = (I + Y^{-1}(I - MY)M^{-1}) = MY$, which concludes the proof.

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**Main Idea:**

If $\mathcal{S}$ is quadratically invariant then $K(I + GK)^{-1} \in \mathcal{S} \Leftrightarrow K \in \mathcal{S}$

# Idea for Numerical Synthesis of a Sparse Controller under QI

**Theorem**

*Given G and a QI sparsity constraint $\mathcal{S}$, there exists a stabilizing K in $\mathcal{S}$ if and only if there exists some DCF of G such that*

$$\text{Pattern}(\tilde{X}\tilde{M}) \leq K^{\text{bin}} \quad \text{or} \quad \text{Pattern}(MX) \leq K^{\text{bin}}. \tag{3}$$

Find some Doubly-Coprime Factorization of $G$

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

that satisfies (3)**!**

# Outline: Numerical Synthesis of a Sparse Controller under QI

## The Youla Parametrization to the Rescue

Start with *any* Doubly-Coprime Factorization of the plant:
$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

then for any Youla parameter $Q$

$$\begin{bmatrix} (Y - Q\tilde{N}) & (X + Q\tilde{M}) \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -(\tilde{X} + MQ) \\ N & (\tilde{Y} - NQ) \end{bmatrix} = I_{n_y+n_u}. \tag{4}$$

is another DCF of the plant $G$ and its associated central controller is given by

$$\begin{aligned} K &= -(\tilde{X} + MQ)(\tilde{Y} - NQ)^{-1} \\ &= (Y - Q\tilde{N})^{-1}(X + Q\tilde{M}) \end{aligned}$$

# Outline: Numerical Synthesis of a Sparse Controller under QI

Start with *any* Doubly-Coprime Factorization of the plant:
$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

find some Youla parameter $Q$ such that for the newly obtained DCF

$$\begin{bmatrix} (Y - Q\tilde{N}) & (X + Q\tilde{M}) \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -(\tilde{X} + MQ) \\ N & (\tilde{Y} - NQ) \end{bmatrix} = I_{n_y+n_u}. \quad (5)$$

for which the following holds:

$$\text{Pattern}\left(MQ\tilde{M} + \tilde{X}\tilde{M}\right) \leq K^{\text{bin}} \quad \text{or} \quad \text{Pattern}\left(MQ\tilde{M} + MX\right) \leq K^{\text{bin}}. \quad (6)$$

# Numerical Synthesis of a Sparse Controller under QI

Start with *any* Doubly-Coprime Factorization of the plant:

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

## Corollary

*Given a plant G and a QI sparsity constraint, G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from* any *DCF of G, there exists a Youla parameter Q such that*

$$\text{Pattern}\left(MQ\tilde{M} + \tilde{X}\tilde{M}\right) \leq K^{\text{bin}} \quad \text{or} \quad \text{Pattern}\left(MQ\tilde{M} + MX\right) \leq K^{\text{bin}}. \qquad (7)$$

# Synthesis of a Sparse Controller as an Exact Model–Matching Problem

Start with *any* Doubly-Coprime Factorization of the plant:

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

**Theorem**

*Given a plant G and a QI sparsity constraint $\mathcal{S}$, G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from* any *DCF of G, there exists a Youla parameter Q such that* $\mathrm{vec}(Q)$ *is a stable solution of the following linear system of TFM equations*

$$\Phi\left(M^T \otimes \tilde{M}\right)\mathrm{vec}(Q) = -\Phi\ \mathrm{vec}\left(\tilde{X}\tilde{M}\right), \qquad (8)$$

*where* $\Phi \stackrel{def}{=} I - diag(K^{\mathrm{bin}})$*. If a stabilizing controller in $\mathcal{S}$ exists then it can be written as* $K = (Y - Q\tilde{N})^{-1}(X + Q\tilde{M})$*.*

# The Exact Model–Matching Problem with Stability

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Exact Model–Matching with Stability:

$$\boxed{\Phi\left(M^T \otimes \tilde{M}\right)\mathrm{vec}(Q) = -\Phi\ \mathrm{vec}\left(\tilde{X}\tilde{M}\right)}$$

- Exact Model–Matching (Wolovich, 1970s )
- Exact Model–Matching with Stability (Antsaklis, 1980s)
- Numerical Methods for Exact Model–Matching with Stability (Chu & Van Dooren, Automatica, 2006.)

# The Youla Parametrization of All Sparse Stabilizing Controllers

Start some *any* Doubly-Coprime Factorization of the plant:

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+p}$$

that satisfies

$$\text{Pattern}(\tilde{X}\tilde{M}) \leq K^{\text{bin}} \quad \text{or} \quad \text{Pattern}(MX) \leq K^{\text{bin}}. \tag{9}$$

## Corollary

*Consider a plant G and a QI sparsity constraint $\mathcal{S}$. If G is stabilizable by a controller K in $\mathcal{S}$, and consequently a DCF of G satisfying (9) exists, the set of all stabilizing controllers of G belonging to the set $\mathcal{S}$ is given by $K = (\tilde{X} + MQ)(\tilde{Y} - NQ)^{-1}$ and the Youla parameter Q must satisfy:*

$$\text{vec}(Q) \in \text{Null}\Big(\Phi\big(M^T \otimes \tilde{M}\big)\Big), \tag{10}$$

*where $\Phi \stackrel{def}{=} I - diag(K^{\text{bin}})$.*

# Sparsity Constrained Model-Matching Problem

## Corollary

*Consider a plant G and a QI sparsity constraint $\mathcal{S}$. If G is stabilizable by a controller K in $\mathcal{S}$, and consequently a DCF of G satisfying (9) exists, the set of all stabilizing controllers of G belonging to the set $\mathcal{S}$ is given by $K = (\tilde{X} + MQ)(\tilde{Y} - NQ)^{-1}$ where the Youla parameter Q must satisfy:*

$$\mathrm{vec}(Q) \in \mathrm{Null}\Big(\Phi\big(M^T \otimes \tilde{M}\big)\Big), \tag{11}$$

*where $\Phi \overset{def}{=} I - diag(K^{\mathrm{bin}})$.*

The sparsity constrained model-matching program is given by:

$$\boxed{\min_{vec(Q)\text{ stab. in Null}\left(\Phi(M^T \otimes \tilde{M})\right)} \|T_1 + T_2 Q T_3\|}$$

# Numerical Example

$$G = \begin{bmatrix} \frac{1}{\lambda+4} & \frac{1}{\lambda-2} \\ \frac{1}{\lambda-1} & 0 \\ \frac{1}{\lambda+5} & \frac{1}{\lambda-3} \end{bmatrix}, \; K^{\text{bin}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We use Nett & Jacobson's state–space formulas to obtain the following DCF of $G$:

$$\tilde{M} = \begin{bmatrix} \frac{\lambda-2}{\lambda+6} & 0 & 0 \\ 0 & \frac{\lambda-1}{\lambda+7} & \frac{\lambda-3}{\lambda+7} \\ 0 & 0 & \frac{\lambda-3}{\lambda+8} \end{bmatrix}, \; M = \begin{bmatrix} \frac{\lambda-1}{\lambda+9} & 0 \\ 0 & \frac{(\lambda-2)(\lambda-3)}{(\lambda+10)(\lambda+11)} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{\lambda-2}{\lambda+6} & \frac{1}{\lambda+7} & \frac{\lambda-3}{\lambda+8} \\ \frac{1}{\lambda+6} & \frac{\lambda-1}{\lambda+7} & \frac{1}{\lambda+8} \end{bmatrix}$$

Need to find $Q$ that satisfies:

$$\Phi\left(M^T \otimes \tilde{M}\right)\mathrm{vec}(Q) = -\Phi\,\mathrm{vec}\left(MX\right), \tag{12}$$

where $\Phi \overset{def}{=} I - diag(K^{\text{bin}})$.

# Numerical Example

$$\tilde{M} = \begin{bmatrix} \frac{\lambda-2}{\lambda+6} & 0 & 0 \\ 0 & \frac{\lambda-1}{\lambda+7} & \frac{\lambda-3}{\lambda+7} \\ 0 & 0 & \frac{\lambda-3}{\lambda+8} \end{bmatrix}, \quad M = \begin{bmatrix} \frac{\lambda-1}{\lambda+9} & 0 \\ 0 & \frac{(\lambda-2)(\lambda-3)}{(\lambda+10)(\lambda+11)} \end{bmatrix}$$

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Need to find $Q$ that satisfies:

$$\Phi\left(M^T \otimes \tilde{M}\right)\mathrm{vec}(Q) = -\Phi\ \mathrm{vec}\left(MX\right), \tag{13}$$

where $\Phi \stackrel{def}{=} I - diag(K^{\mathrm{bin}})$. In this case a solution can be found to be:

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{\lambda+8}{\lambda+7} \end{bmatrix}$$

The resulting stabilizing central controller is given by:

$$K = \begin{bmatrix} \frac{\lambda+17}{\lambda+7} & 0 \\ 754\,\frac{(\lambda+5.87)(\lambda-0.4525)}{(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+8)} & \frac{(\lambda+42.5389)(\lambda-2.5389)}{(\lambda+6)(\lambda+8)} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \frac{1}{\lambda+7} & 0 \\ \frac{1}{\lambda+6} & 0 & \frac{1}{\lambda+8} \end{bmatrix},$$

# A Meaningful, Particular Case

Decoupled Doubly-Coprime Factorizations

# Preliminary notation and definitions:

The input and output are partitioned as follows:

$$
\begin{aligned}
y^T &= \left[y_{[1]}^T \cdots y_{[r_y]}^T\right]^T, \qquad \textstyle\sum_{i=1}^{r_y} m_i &= m \\
u^T &= \left[u_{[1]}^T \cdots u_{[r_u]}^T\right]^T, \qquad \textstyle\sum_{i=1}^{r_u} p_i &= p
\end{aligned}
\tag{14}
$$

The partition above induces the following block-partition of $G$ and $K$:

$$
\begin{aligned}
G &= \begin{bmatrix} G_{[11]} & \cdots & G_{[1\,r_u]} \\ \vdots & \ddots & \vdots \\ G_{[r_y\,1]} & \cdots & G_{[r_y\,r_u]} \end{bmatrix} \\
K &= \begin{bmatrix} K_{[11]} & \cdots & K_{[1\,r_y]} \\ \vdots & \ddots & \vdots \\ K_{[r_u\,1]} & \cdots & K_{[r_u\,r_y]} \end{bmatrix}
\end{aligned}
\tag{15}
$$

# Preliminary notation and definitions:

## Remark

*Given factorizations of G and K as $G = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ and $K = \tilde{X}\tilde{Y}^{-1} = Y^{-1}X$, respectively, the partition in (14) will induce a unique block-partition structure on the factors $N$, $M$, $\tilde{N}$, $\tilde{M}$, $X$, $Y$, $\tilde{X}$ and $\tilde{Y}$ as well.*

## Definition

Let $\tilde{N}$ and $\tilde{M}$ be a factorization of $G$. The pair $(\tilde{N}, \tilde{M})$ is called output decoupled if $\tilde{M}$ has the following block diagonal structure:

$$\tilde{M} = \text{diag}(\{\tilde{M}_{[ii]}\}_{i=1}^{r_y}) \tag{16}$$

where $\text{diag}(\{M_{[ii]}\}_{i=1}^{r_y})$ is defined as:

$$\text{diag}(\{\tilde{M}_{[ii]}\}_{i=1}^{r_y}) \overset{def}{=} \begin{bmatrix} \tilde{M}_{[11]} & 0 & \cdots & 0 \\ 0 & \tilde{M}_{[22]} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{M}_{[r_y r_y]} \end{bmatrix} \tag{17}$$

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## Remark

*Notice that an output (input) decoupled factorization can always be constructed by factoring each block row of G separately as follows:*

$$\left[G_{[i1]} \cdots G_{[ir_u]}\right] = \tilde{M}_{[ii]}^{-1} \left[\tilde{N}_{[i1]} \cdots \tilde{N}_{[ir_u]}\right], \qquad i \in \overline{1, r_y} \tag{18}$$

# Preliminary notation and definitions:

## Definition

Let $\tilde{N}$ and $\tilde{M}$ be a factorization of $G$. The pair $(\tilde{N}, \tilde{M})$ is called output decoupled if $\tilde{M}$ has the following block diagonal structure:

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## Definition

A DCF $(M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y})$ of $G$ is called input/output decoupled if the pairs $(N, M)$ and $(\tilde{N}, \tilde{M})$ are input and output decoupled, respectively.

# On Input/Output DCFs

Input/output decoupled pairs may not be coprime, much less doubly coprime

$$\left[G_{[i1]} \cdots G_{[ir_u]}\right] = \tilde{M}_{[ii]}^{-1} \left[\tilde{N}_{[i1]} \cdots \tilde{N}_{[ir_u]}\right], \qquad i \in \overline{1, r_y}. \tag{18}$$

$$\begin{bmatrix} G_{[11]} & \cdots & G_{[1r_u]} \\ \vdots & \ddots & \vdots \\ G_{[r_y1]} & \cdots & G_{[r_yr_u]} \end{bmatrix} = \begin{bmatrix} \tilde{M}_{[11]}^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{M}_{[r_yr_y]}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{N}_{[11]} & \cdots & \tilde{N}_{[1r_u]} \\ \vdots & \ddots & \vdots \\ N_{[r_y1]} & \cdots & \tilde{N}_{[r_yr_u]} \end{bmatrix} \tag{19}$$

## Proposition

*The output decoupled factorization (19) is coprime if and only if the TFM*

$$\Psi = \begin{bmatrix} \tilde{M}_{[11]} & \cdots & 0 & \tilde{N}_{[11]} & \cdots & \tilde{N}_{[1r_u]} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \tilde{M}_{[r_yr_y]} & N_{[r_y1]} & \cdots & \tilde{N}_{[r_yr_u]} \end{bmatrix}$$

*has full row–rank at the finite set of points $\lambda \in (\mathbb{C} - \Omega)$ (unstable poles of G).*

# Preliminary notation and definitions:

## Definition

Given $K$ in $\mathbb{R}(\lambda)^{p\times m}$, we define $\mathrm{Pattern}(K) \in \mathbb{B}^{r_u\times r_y}$ as follows:

$$\mathrm{Pattern}(K)_{ij} \overset{def}{=} \begin{cases} 0 & \text{if} \quad K_{[ij]} = 0_{p_i\times m_j} \\ 1 & \text{otherwise} \end{cases} \qquad i,j \in \overline{1,r_u} \times \overline{1,r_y} \tag{20}$$

where $0_{p_i\times m_j}$ is a matrix with $p_i$ rows and $m_j$ columns and whose entries are all zero.

## Definition

Conversely, for any binary matrix $K^{\mathrm{bin}}$ in $\mathbb{B}^{r_u\times r_y}$, we define the following linear subspace:

$$\mathrm{Sparse}(K^{\mathrm{bin}}) \overset{def}{=} \left\{ K \in \mathbb{R}(\lambda)^{p\times m} \middle| \mathrm{Pattern}(K) \leq K^{\mathrm{bin}} \right\} \tag{21}$$

## Definition

Given $K^{\mathrm{bin}}$ in $\mathbb{B}^{r_u\times r_y}$, the *sparsity constraint* $\mathcal{S}$ is defined as:

$$\mathcal{S} \overset{def}{=} Sparse(K^{\mathrm{bin}}), \tag{22}$$

# Preliminary Facts:

## Remark

*As a consequence of the definitions above, the following holds for any input/output decoupled DCF of G:*

$$\begin{array}{lclclcl} Pattern(M) & \leq & I_{r_u \times r_u}, & \quad & Pattern(N) & \leq & G^{bin} \\ Pattern(\tilde{M}) & \leq & I_{r_y \times r_y}, & \quad & Pattern(\tilde{N}) & \leq & G^{bin} \end{array}$$

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Recall the following Theorem ...

## Theorem

*Given a plant G and a QI sparsity constraint, G is stabilizable with a sparsity constrained controller K belonging to the set $\mathcal{S}$ if and only if, starting from* any *DCF of G, there exists a Youla parameter Q such that* $\mathrm{vec}(Q)$ *is a stable solution to the linear system of TFM equations*

$$\Phi\left(M^T \otimes \tilde{M}\right)\mathrm{vec}(Q) = -\Phi \,\mathrm{vec}\left(\tilde{X}\tilde{M}\right)$$

*where* $\Phi \overset{def}{=} I - diag(K^{\mathrm{bin}})$.

# Specialized Results for the Input/Output Decoupled DCF Case:

## Remark

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From Previous Theorem ...

$$\Phi\left(M^T \otimes \tilde{M}\right)\mathrm{vec}(Q_0) = -\Phi\ \mathrm{vec}\left(\tilde{X}\tilde{M}\right)$$

## Corollary

*Let $\mathcal{S}$ be a given QI sparsity constraint and $(M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y})$ an input/output decoupled DCF of G. Assume that there is a stabilizing controller in $\mathcal{S}$ and let stable $Q_0$ be selected to satisfy the condition above. Any stabilizing controller in $\mathcal{S}$ can be written as $K = (Y - Q\tilde{N})^{-1}(X + Q\tilde{M})$, where Q is obtained as:*

$$Q = Q_0 + Q_\delta, \qquad \text{stable } Q_\delta \in \mathcal{S} \tag{23}$$

# Specialized Results for the Input/Output Decoupled DCF Case:

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From Previous Theorem ...

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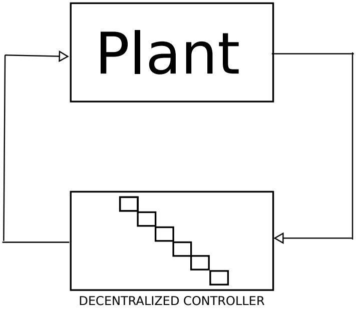
## Corollary

*Let $(M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y})$ be an input/output decoupled DCF of G. Given a QI sparsity constraint $\mathcal{S}$, G is stabilizable by a controller in $\mathcal{S}$ if and only if $M^{-1}\tilde{X}_{\mathcal{S}_\perp}$ is stable, where $\tilde{X}_{\mathcal{S}_\perp}$ results from the additive factorization $\tilde{X} = \tilde{X}_{\mathcal{S}} + \tilde{X}_{\mathcal{S}_\perp}$ satisfying $\mathrm{Pattern}(\tilde{X}_{\mathcal{S}}) \leq K^{\mathrm{bin}}$ and $\mathrm{Pattern}(\tilde{X}_{\mathcal{S}_\perp}) \leq K_\perp^{\mathrm{bin}}$.*

# Theme 2




*classical*

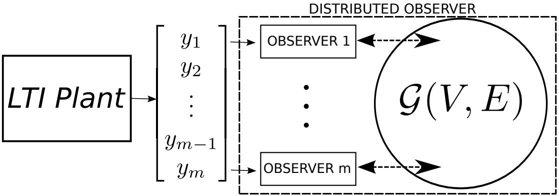
Characterization of stabilizability:

- Fixed Modes,
- Structured fixed modes,
- Quotient fixed modes

*new applications*

✡ Distributed State Estimation

# Problem formulation



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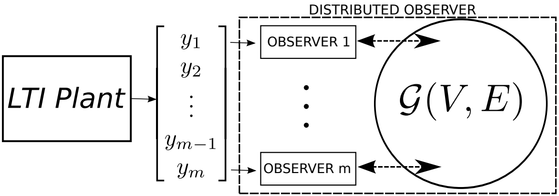


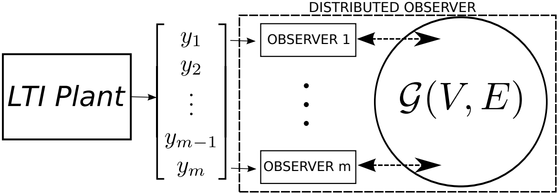

LTI plant is described as:

$$x(k+1) = Ax(k)$$
$$y(k) = Cx(k)$$

where $y(k) = \left(y_1^T(k), \cdots, y_m^T(k)\right)^T$

$$x(k) \in \mathbb{R}^n, y_i(k) \in \mathbb{R}^{r_i}$$

# Problem formulation




**Definition:** Consider a LTI plant with state $x(k)$ and a distributed observer with local state estimates $\{\hat{x}_i(k)\}_{i \in V}$. The distributed observer is said to achieve *omniscience* asymptotically if the following holds:

$$\lim_{k \to \infty} ||\hat{x}_i(k) - x(k)|| = 0, i \in V$$

**Objective** Determine whether a LTI distributed observer exists for which omniscience is attained, and if so construct one.

**Advantages:** Tractable robustness analysis and frequency domain performance analysis in the presence of exogenous inputs

# Related Work


- R. Olfati-Saber, J.A. Fax, and R.M. Murray, Consensus and cooperation in networked multi-agent systems, Proceedings of the IEEE, vol. 95, no. 1, Jan. 2007.
- R. Carli, A. Chiuso, L. Schenato, and S. Zampieri, Distributed kalman filtering based on consensus strategies, IEEE Journal on Selected Areas in Communication, no. 4, May 2008.
- U. A. Khan and A. Jadbabaie, On the stability and optimality of distributed kalman filters with finite-time data fusion, in 2011 American Control Conference, June 2011.
- P. Alriksson and A. Rantzer, Distributed kalman filtering using weighted averaging, in In Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, 2006.

# A convenient distributed observer structure

Consider the following structure for each ”local” observer:

$$\hat{x}_i(k+1) = A \sum_{j \in \mathcal{N}_i} \mathbf{w}_{ij} \underbrace{\hat{x}_j(k)}_{\text{state estimate}} + \mathbf{H}_i \underbrace{(y_i(k) - C_i \hat{x}_i(k))}_{\text{measurement innovation}} + \mathbf{Q}_i \underbrace{z_i(k)}_{\text{aug. state}}, \quad i \in V$$

$$z_i(k+1) = \mathbf{R}_i (y_i(k) - C_i \hat{x}_i(k)) + \mathbf{S}_i z_i(k)$$

where $\mathbf{H}_i \in \mathbb{R}^{n \times r_i}$, $\mathbf{Q}_i \in \mathbb{R}^{n \times \mu_i}$, $\mathbf{R}_i \in \mathbb{R}^{\mu_i \times r_i}$, $\mathbf{S}_i \in \mathbb{R}^{\mu_i \times \mu_i}$, and $\mu_i$ is the dimension of $z_i(k)$. We refer to $\mathbf{W} = (\mathbf{w}_{ij})_{i,j \in V}$ as a weight matrix, and $\{\mathbf{H}_i, \mathbf{Q}_i, \mathbf{R}_i, \mathbf{S}_i\}$ as gain matrices. The neighborhood $\mathcal{N}_i$ consists of the vertices with outgoing edges terminating in $i$. These matrices are the design parameters that need to be computed.

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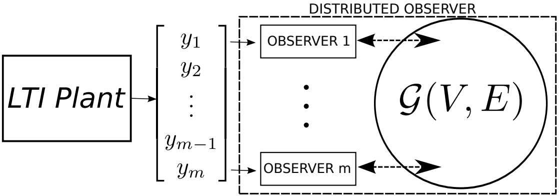
Notice that the network is used to disseminate state estimates.

# Main result




**Theorem:** Let $\mathcal{G}(V, E)$ be a strongly connected communication graph. There exist a stochastic weight matrix $\mathbf{W} = (\mathbf{w}_{ij})_{i,j \in V}$ and gain matrices $\{\mathbf{H}_i, \mathbf{Q}_i, \mathbf{R}_i, \mathbf{S}_i\}_{i \in V}$ such that the resulting distributed observer achieves omniscience asymptotically if and only if the pair $(A, C)$ is detectable.

# Key observation and connections to decentralized stabilization and fixed modes

Notice that we can write the error dynamics of as follows.

$$\epsilon_i(k+1) = \sum_{j \in \mathcal{N}_i} \mathbf{w}_{ij} A \epsilon_j(k) - \mathbf{H}_i C_i \epsilon_i(k) - \mathbf{Q}_i z_i(k)$$

$$z_i(k+1) = \mathbf{R}_i C_i \epsilon_i(k) + \mathbf{S}_i z_i(k)$$

where $\epsilon_i(k) \stackrel{def}{=} x(k) - \hat{x}_i(k)$. We can also write as follows :

$$\begin{pmatrix} \epsilon(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} \mathbf{W} \otimes A - \bar{B}\bar{\mathbf{H}}\bar{C} & -\bar{B}\bar{\mathbf{Q}} \\ \bar{\mathbf{R}}\bar{C} & \bar{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \epsilon(k) \\ z(k) \end{pmatrix}$$

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$$\begin{pmatrix} \epsilon(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} \mathbf{W} \otimes A - \bar{B}\bar{\mathbf{H}}\bar{C} & -\bar{B}\bar{\mathbf{Q}} \\ \bar{\mathbf{R}}\bar{C} & \bar{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \epsilon(k) \\ z(k) \end{pmatrix}$$

where

$$\bar{B} = (\bar{B}_1, \cdots, \bar{B}_m) \text{ with } \bar{B}_i = e_m^{(i)} \otimes I_n$$

$$\bar{C} = \left(\bar{C}_1^T, \cdots, \bar{C}_m^T\right)^T \text{ with } \bar{C}_i = \left(e_m^{(i)}\right)^T \otimes C_i$$

$$\bar{\mathbf{H}} = diag\,(\mathbf{H}_1, \cdots, \mathbf{H}_m)\,, \quad \bar{\mathbf{Q}} = diag\,(\mathbf{Q}_1, \cdots, \mathbf{Q}_m)$$

$$\bar{\mathbf{R}} = diag\,(\mathbf{R}_1, \cdots, \mathbf{R}_m)\,, \quad \bar{\mathbf{S}} = diag\,(\mathbf{S}_1, \cdots, \mathbf{S}_m)$$

# Key observation and connections to decentralized stabilization and fixed modes

Notice that we can write the error dynamics of as follows.

$$\epsilon_i(k+1) = \sum_{j \in \mathcal{N}_i} \mathbf{w}_{ij} A \epsilon_j(k) - \mathbf{H}_i C_i \epsilon_i(k) - \mathbf{Q}_i z_i(k)$$

$$z_i(k+1) = \mathbf{R}_i C_i \epsilon_i(k) + \mathbf{S}_i z_i(k)$$

where $\epsilon_i(k) \stackrel{def}{=} x(k) - \hat{x}_i(k)$. We can also write as follows :

$$\begin{pmatrix} \epsilon(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} \mathbf{W} \otimes A - \bar{B}\bar{\mathbf{H}}\bar{C} & -\bar{B}\bar{\mathbf{Q}} \\ \bar{\mathbf{R}}\bar{C} & \bar{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \epsilon(k) \\ z(k) \end{pmatrix}$$

Necessary and sufficient conditions for stabilizability as well as design methods have been proposed in:

- S.-H. Wang and E. J. Davison, On the stabilization of decentralized control systems, IEEE Trans. Automat. Contr., vol. AC-18, no. 5, Oct. 1973.
- B. D. O. Anderson and D. J. Clements, Algebraic characterization of fixed modes in decentralized control, Automatica, vol. 17, no. 5, pp. 703712, 1981.
- E. J. Davison and U. Ozguner, Characterizations of decentralized fixed modes for interconnected systems, Automatica, vol. 19, no. 2, pp. 169182, 1983.

# Conclusions and Open Questions

- New recent results have provided algebraic techniques to test the existence of convex parametrizations of sparsity-constrained controllers.
- We have leveraged on these recent ideas to develop a factorization-based theory that extends Youla's classical formulation for the design of sparsity constrained controllers. The key idea is recasting the sparsity constraints on the controller as subspace constraints (hence convex) on the Youla parameter.
- We are currently working on a simple method to optimally modify one block of an existing stabilizing block diagonal controller. There are no results on effective independent search methods, with performance guarantees.

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