

Democracy and the role of minorities in Markov chain models

Non-reversible perturbations of Markov chains models

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Perturbation of dynamical networks

The stability of a complex large-scale dynamical network under localized perturbations is one of the paradigmatic problem of these decades.

Key issues:

- ▶ **Correlation:** understand how local perturbation affect the overall behavior.
- ▶ **Resilience** find bounds on the perturbation 'size' which the network can tolerate.
- ▶ **Phase transitions**

Perturbation of dynamical networks

State of the art:

- ▶ Most of the results available in the literature are on connectivity issues.
- ▶ Analysis of how the perturbation is altering the degree distribution of the network.
- ▶ Degrees are in general not sufficient to study dynamics.
- ▶ Example: non-reversible Markov chain models.

Perturbation of dynamical networks

What type of perturbations:

- ▶ Failures in nodes or links in sensor or computer networks. Sensor with different technical properties.
- ▶ Heterogeneity in opinion dynamics models: minorities, leaders exhibiting a different behavior
- ▶ A subset of control nodes in the network...

In this talk:

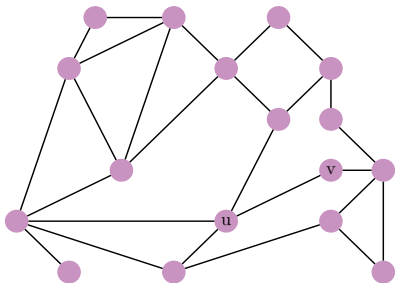
- ▶ Non-reversible perturbations of Markov chain models
- ▶ Applications to consensus dynamics

Outline

- ▶ Perturbation of consensus dynamics.
- ▶ The general setting: perturbation of Markov chain models.
- ▶ An example: heterogeneous gossip model.
- ▶ Results on how the perturbation is affecting the asymptotics.
- ▶ Conclusions and open issues.

Consensus dynamics

$G = (V, E)$ connected graph



y_v initial state (opinion) of node v

Dynamics: $y(t+1) = Py(t)$, $y(0) = y$

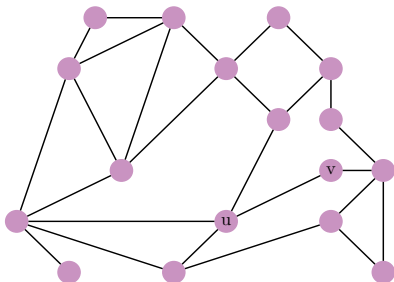
$P \in \mathbb{R}^{V \times V}$ stochastic matrix on G ($P_{uv} > 0 \Leftrightarrow (u, v) \in E$)

Consensus: $\lim_{t \rightarrow +\infty} (P^t y)_u = \pi^* y$ for all u (* means transpose)

$\pi \in \mathbb{R}_+^V$, $\pi^* P = \pi^*$, $\sum_u \pi_u = 1$ (invariant probability)

Consensus dynamics

$G = (V, E)$ connected graph



Example: (SRW) $P_{uv} = \frac{1}{d_u}$, d_u degree of node u

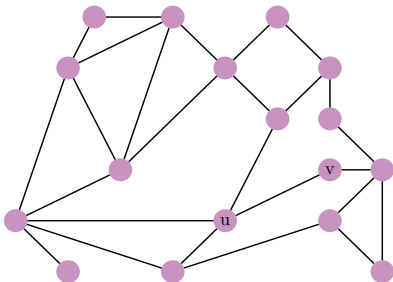
Explicit expression for π : $\pi_u = \frac{d_u}{2|E|}$

π essentially depends on local properties of G .

This holds true for general reversible Markov chains.

Consensus dynamics

$G = (V, E)$ connected graph



Dynamics: $y(t+1) = Py(t)$, $y(0) = y$

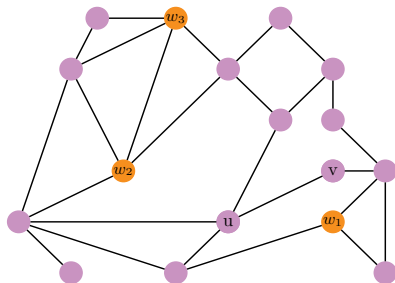
Consensus: $\lim_{t \rightarrow +\infty} (P^t y)_u = \pi^* y$ for all u

Two important parameters:

- ▶ the **invariant probability** π responsible for the **asymptotics**
- ▶ the **mixing time** τ responsible for the **transient** behavior (speed of convergence)

Perturbation of consensus dynamics

$G = (V, E)$ connected graph



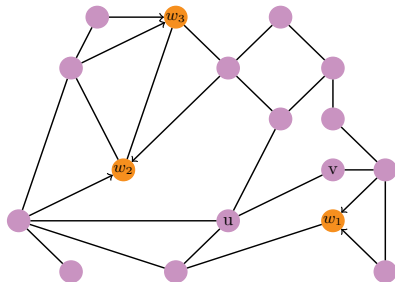
► $P \in \mathbb{R}^{V \times V}$ stochastic matrix on G

► Perturb P in a small set of nodes:

$$\tilde{P}_{uv} = P_{uv} \text{ if } u \notin W = \{w_1, w_2, w_3\}.$$

Perturbation of consensus dynamics

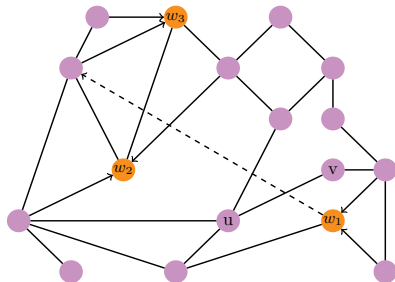
$G = (V, E)$ connected graph



- ▶ $P \in \mathbb{R}^{V \times V}$ stochastic matrix on G
- ▶ Perturb P in a small set of nodes:
 $\tilde{P}_{uv} = P_{uv}$ if $u \notin W = \{w_1, w_2, w_3\}$.
- ▶ Cut edges

Perturbation of consensus dynamics

$G = (V, E)$ connected graph



► $P \in \mathbb{R}^{V \times V}$ stochastic matrix on G

► Perturb P in a small set of nodes:

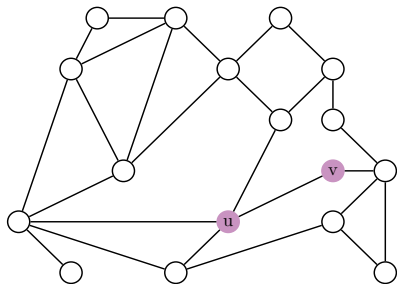
$$\tilde{P}_{uv} = P_{uv} \text{ if } u \notin W = \{w_1, w_2, w_3\}.$$

► Cut edges. Add new edges.

A heterogeneous gossip model

(Acemoglu et al. 2009)

$G = (V, E)$, $W \subset V$ a minority of **influential** (stubborn) agents

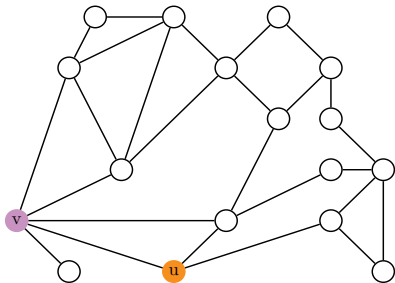


- ▶ At each time t choose an edge $\{u, v\}$ at random.
- ▶ If $u, v \in V \setminus W$,
 $y_u(t+1) = y_v(t+1) = (x_u(t) + x_v(t))/2$ (reg. interaction)

A heterogeneous gossip model

(Acemoglu et al. 2009)

$G = (V, E)$, $W \subset V$ a minority of **influential** (stubborn) agents

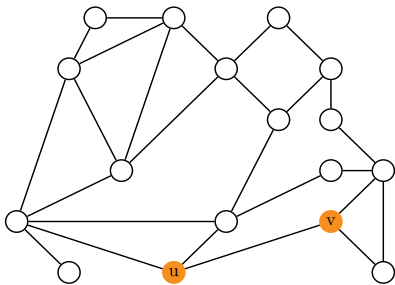


- ▶ At each time t choose an edge $\{u, v\}$ at random.
- ▶ If $u \in W$, $v \in V \setminus W$,
 - ▶ $y_u(t+1) = y_u(t)$, $y_v(t+1) = \varepsilon y_v(t) + (1 - \varepsilon)y_u(t)$
with probability p (**forceful interaction**)
 - ▶ $y_u(t+1) = y_v(t+1) = (x_u(t) + x_v(t))/2$
with probability $1 - p$ (reg. interaction)

A heterogeneous gossip model

(Acemoglu et al. 2009)

$G = (V, E)$, $W \subset V$ a minority of **influential** (stubborn) agents



- ▶ At each time t choose an edge $\{u, v\}$ at random.
- ▶ If $u, v \in W$ nothing happens.

A heterogeneous gossip model

- ▶ $y(t+1) = P(t)y(t)$
- ▶ $y(t)$ converges to a consensus almost surely if $p \in [0, 1)$.
But what type of consensus?
- ▶ If no forceful interaction is present ($p = 0$),
 $y(t)_u \rightarrow N^{-1} \sum_v y(0)_v$ for every u .
- ▶ $\mathbb{E}(P(t)) = P_p$
- ▶ P_p and P_0 only differ in the rows having index in $W \cup \partial W$.

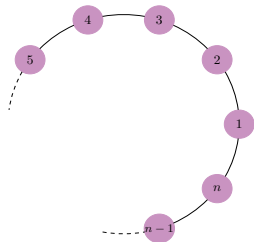
Perturbation of Markov chain models

The abstract setting:

- ▶ $G = (V, E)$ family of connected graphs. $N = |V| \rightarrow +\infty$.
- ▶ $W \subseteq V$ perturbation set
- ▶ P, \tilde{P} stochastic matrices on G . $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- ▶ $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.
- ▶ Study $\|\pi - \tilde{\pi}\|_{\text{TV}} := \frac{1}{2} \sum_v |\pi_v - \tilde{\pi}_v|$ (as a function of N)
Notice that $|\tilde{\pi}^* y - \pi^* y| \leq \|\pi - \tilde{\pi}\|_{\text{TV}} \|y\|_{\infty}$

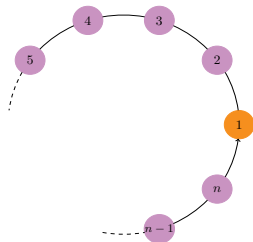
The ideal result: $\pi(W) \rightarrow 0 \Rightarrow \|\tilde{\pi} - \pi\|_{\text{TV}} \rightarrow 0$

A counterexample



$$P_{u,u+1} = P_{u,u-1} = 1/2, \quad \pi \text{ uniform}$$

A counterexample



$$P_{u,u+1} = P_{u,u-1} = 1/2, \quad \pi \text{ uniform}$$

$$\tilde{P}_{1,2} = 1, \quad \tilde{P}_{1,n} = 0, \quad \tilde{\pi}_1 = 1/n, \quad \tilde{\pi}_j = \frac{2(n-j+1)}{n^2} \text{ for } j \geq 1$$

$$\|\pi - \tilde{\pi}\|_{\text{TV}} \asymp \text{cost.}$$

Perturbation of Markov chain models

- ▶ $G = (V, E)$ family of connected graphs. $N = |V| \rightarrow +\infty$.
- ▶ $W \subseteq V$ perturbation set
- ▶ P, \tilde{P} stochastic matrices on G . $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- ▶ $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

If the chain mixes slowly, the process will pass many times through the perturbed set W before getting to equilibrium. $\tilde{\pi}$ will be largely influenced by the perturbed part.

Consequence: $\|\pi - \tilde{\pi}\|_{\text{TV}} \not\rightarrow 0$

Perturbation of Markov chain models

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- ▶ P, \tilde{P} stochastic matrices on G . $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- ▶ $\pi^* P = \pi^*, \tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

A more realistic result:

P mixes suff. fast, $\pi(W) \rightarrow 0 \Rightarrow \tilde{\pi} - \pi \rightarrow 0$

Recall: mixing time $\tau := \min\{t \mid \|\mu^* P^t - \pi^*\|_{\text{TV}} \leq 1/e \forall \mu\}$

SRW on d -grid with N nodes, $\tau \asymp N^{2/d} \ln N$

SRW on Erdos-Renji, small world, configuration model $\tau \asymp \ln N$

Perturbation of Markov chain models: the literature

- ▶ $G = (V, E)$ family of connected graphs. $N = |V| \rightarrow +\infty$.
- ▶ $W \subseteq V$ perturbation set
- ▶ P, \tilde{P} stochastic matrices on G . $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- ▶ $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

$$\|\tilde{\pi} - \pi\|_{\text{TV}} \leq C_{\tau} \|\tilde{P} - P\|_1 \quad (\text{Mitrophanov, 2003})$$

To measure perturbations of P , the 1-norm is not good to treat localized perturbations: if P and \tilde{P} differ just in one row u and $|P_{uv} - \tilde{P}_{uv}| = \delta$, then, $\|P - \tilde{P}\|_1 \geq \delta$ and will not go to 0 for $N \rightarrow \infty$. In our context, the bound will always blow up.

Perturbation of Markov chain models

- ▶ $G = (V, E)$ family of connected graphs. $N = |V| \rightarrow +\infty$.
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- ▶ $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

A more realistic result:

P mixes suff. fast, $\pi(W) \rightarrow 0 \Rightarrow \tilde{\pi} - \pi \rightarrow 0$

There is another problem: if P mixes rapidly, nobody guarantees that \tilde{P} will also do...

Perturbation of Markov chain models: first result

- ▶ $G = (V, E)$ family of connected graphs. $N = |V| \rightarrow +\infty$.
- ▶ $W \subseteq V$ perturbation set
- ▶ P, \tilde{P} stochastic matrices on G . $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- ▶ $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

Theorem

$$\|\tilde{\pi} - \pi\|_{TV} \leq \tau \tilde{\pi}(W) \log \frac{e^2}{\tau \tilde{\pi}(W)}. \quad (1)$$

or, symmetrically,

$$\|\tilde{\pi} - \pi\|_{TV} \leq \tilde{\tau} \pi(W) \log \frac{e^2}{\tilde{\tau} \pi(W)}. \quad (2)$$

Proof: Coupling technique.

Perturbation of Markov chain models: first result

Corollary

$$\tau\pi(W) \rightarrow 0, \tilde{\tau} = O(\tau) \Rightarrow \|\tilde{\pi} - \pi\|_{TV} \rightarrow 0$$

$$\tau\pi(W) \rightarrow 0, \tilde{\pi}(W) = O(\pi(W)) \Rightarrow \|\tilde{\pi} - \pi\|_{TV} \rightarrow 0$$

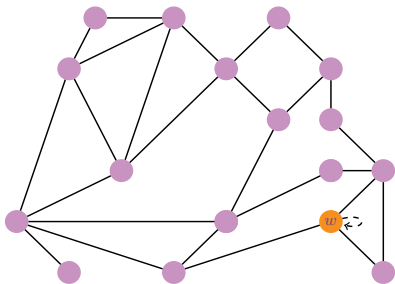
The perturbation, in order to achieve a modification of the invariant probability, necessarily has to

- ▶ slow down the chain
- ▶ increase the probability on the perturbation subset W .

π and τ are intimately connected to each other!

Perturbation of Markov chain models: first result

Slowing down the chain and putting weight on W look quite connected to each other and essentially amounts to decrease the probability of exiting W :



$$\tilde{P}_{ww} = 1 - 1/N$$

$$\tilde{\pi}_w = \mathbb{E}(\tilde{T}_w^+)^{-1} = \frac{1}{1 - N^{-1} + N^{-1}\mathbb{E}(T_w^+)} = \frac{\pi_w}{(1 - N^{-1})\pi_w + N^{-1}}$$

$$\pi_w \sim \frac{k}{N} \Rightarrow \tilde{\pi}_w \sim \frac{k}{k+1}$$

A deeper analysis

Lemma

$$\tilde{\pi}(W) \leq \frac{1}{1 + \tilde{\phi}_W^* \tau_W^*},$$

where

$$\tau_W^* := \min\{\mathbb{E}_v[T_W] : v \in V \setminus W\},$$

minimum entrance time to W

it depends on P

$$\tilde{\phi}_W^* := \frac{\sum_{w \in W} \sum_{v \in V \setminus W} \tilde{\pi}_w \tilde{P}_{vw}}{\tilde{\pi}(W)}$$

bottleneck ratio of W

it depends on \tilde{P}

Proof From Kac's lemma

$$\tilde{\pi}(W)^{-1} = \mathbb{E}_{\tilde{\pi}_w}[T_W^+] = 1 + \sum_w \sum_v \frac{\tilde{\pi}_w}{\tilde{\pi}(W)} \tilde{P}_{vw} \mathbb{E}_v[T_W] \geq 1 + \tilde{\phi}_W^* \tau_W^*.$$



A deeper analysis

- ▶ **bottleneck ratio** \longleftrightarrow exit probability from W :

$$\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha \quad \forall w \in W \Rightarrow \tilde{\phi}_W^* \geq d/\alpha$$

If $\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha$ for fixed $d, \alpha > 0$ and for every $w \in W$, then

$$\tilde{\pi}(W) \leq \frac{1}{1 + \tilde{\phi}_W^* \tau_W^*} \asymp (\tau_W^*)^{-1}, \quad \tau \tilde{\pi}(W) = O\left(\frac{\tau}{\tau_W^*}\right)$$

Hence,

$$\frac{\tau}{\tau_W^*} \rightarrow 0 \Rightarrow \|\tilde{\pi} - \pi\|_{TV} \rightarrow 0$$

A deeper analysis

- ▶ minimum entrance time $\longleftrightarrow \pi(W)$:

$$\tau_W^* := \min\{\mathbb{E}_v[T_W]\} \asymp \pi(W)^{-1} \text{ (Conjecture)}$$

$$\text{(Kac's lemma: } \pi(W)^{-1} = 1 + \sum_w \sum_v \frac{\pi_w}{\pi(W)} P_{wv} \mathbb{E}_v[T_W]$$

$$\Rightarrow \mathbb{E}_v[T_W] \asymp \pi(W)^{-1} \text{ for some } v, \dots)$$

$\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha$ for every $w \in W$ plus conjecture imply

$$\frac{\tau}{\tau_W^*} \asymp \tau \pi(W) \rightarrow 0 \Rightarrow \|\tilde{\pi} - \pi\|_{TV} \rightarrow 0$$

Examples

The conjecture

$$\tau_W^* := \min\{\mathbb{E}_v[T_W]\} \asymp \pi(W)^{-1}$$

holds if P is the simple random walk SRW on

- ▶ d -grids with $d \geq 3$, $|W|$ bounded.
- ▶ Erdos-Renji, configuration model (w.p. 1) if $|W| = o(N^{1-\epsilon})$

(techniques: electrical network interpretation, effective resistance; locally tree-like graphs)

Recall that

- ▶ d -grid, $\tau \asymp N^{d/2} \ln N$
- ▶ Erdos-Renji, configuration model (w.p. 1) $\tau = O(\ln N)$

Examples

Theorem

- ▶ $G = (V, E)$ family of connected graphs. $N = |V| \rightarrow +\infty$.
- ▶ $W \subseteq V$ perturbation set
- ▶ P SRW on G , $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- ▶ $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.
- ▶ $\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha$ for fixed $d, \alpha > 0$ and for every $w \in W$,

If G and W are:

- ▶ d -grids with $d \geq 3$, $|W|$ bounded
- ▶ Erdos-Renji, configuration model (w.p. 1), $|W| = o(N^{1-\epsilon})$

then,

$$\|\tilde{\pi} - \pi\|_{TV} \rightarrow 0$$

Application to the heterogeneous gossip model

- ▶ $G = (V, E)$, $W \subseteq V$ **forceful** agents (with prob. p).
- ▶ $y(t+1) = P(t)y(t)$, $\mathbb{E}(P(t)) = P_p$
- ▶ P_p and P_0 only differ in the rows having index in $W \cup \partial W$.

A specific example: **d -regular (toroidal) grid**.

$P_0 = (1 - N^{-1})\text{Id} + N^{-1}d^{-1}A_G$ is a lazy simple random walk,

π_0 uniform probability, $\tau_0 \asymp N^{2/d+1} \ln N$, $\tau_W^* \asymp \frac{|W|}{N^2}$

$\frac{\tau_0}{\tau_W^*} \asymp |W|N^{2/d-1} \ln N \rightarrow 0$ if $d \geq 3$, $|W|$ bounded.

$$\|\pi_p - \pi_0\|_{\text{TV}} \rightarrow 0$$

- ▶ the **minority** has a **vanishing effect** on the global population
- ▶ $\max_v (\pi_p)_v \rightarrow 0$ **democracy is preserved** ('wise society' in Jackson's terminology)

Gossip with stubborn agents

Take $p = 1$ in the heterogeneous gossip model.

$G = (V, E)$, $W \subset V$ a minority of **influential** (stubborn) agents

- ▶ At each time t choose an edge $\{u, v\}$ at random.
- ▶ If $u, v \in V \setminus W$,

$$y_u(t+1) = y_v(t+1) = (x_u(t) + x_v(t))/2$$

- ▶ If $u \in W$, $v \in V \setminus W$,

$$y_u(t+1) = y_u(t), y_v(t+1) = (y_v(t) + y_u(t))/2$$

Gossip with stubborn agents

(Acemoglu, Como, F., Ozdaglar)

- ▶ $y(t) \rightarrow y(\infty)$ in distribution. ($y_w(\infty) = y_w(0) \forall w \in W$)
- ▶ If $\exists w, w' \in W : y_w(0) \neq y_{w'}(0)$, then,
 $\mathbb{P}(y_v(\infty) \neq y_{v'}(\infty)) > 0$ (asymptotic disagreement)
- ▶ However $\frac{1}{n} \left| \left\{ v : \left| \mathbb{E}[y_v(\infty)] - \xi \right| \geq \varepsilon \right\} \right| \leq C_\varepsilon \tau \pi(W)$
If $\tau \pi(W) \rightarrow 0$, then approximate consensus!

This can also be read as a sort of **lack of controllability**: constraints on the shape of the final state configuration achievable by the global system.

Conclusions and open issues

- ▶ Perturbations of Markov chain models and their effect on the invariant probabilities.
- ▶ If the mixing time is sufficiently small w.r. to the size of the perturbation, the effect on the invariant probability becomes negligible in the large scale limit.
- ▶ Applications to consensus dynamics
- ▶ Find more general estimation of the minimum entrance time parameter τ_W^* .
- ▶ Find estimation of type $c_1 \leq \tilde{\pi}_v / \pi_v \leq c_2$. They would permit to obtain estimations of $|\tilde{\tau} - \tau|$.
- ▶ Study phase transitions.
- ▶ Consider perturbations of non linear models (consensus versus epidemic, threshold models).