Consensus theory and Hilbert metric

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Consensus and coordination on nonlinear spaces (circle, orthogonal group, SE(2), SE(3), ...)

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Geometry and Symmetries in Coordination Control
Alain Sarlette
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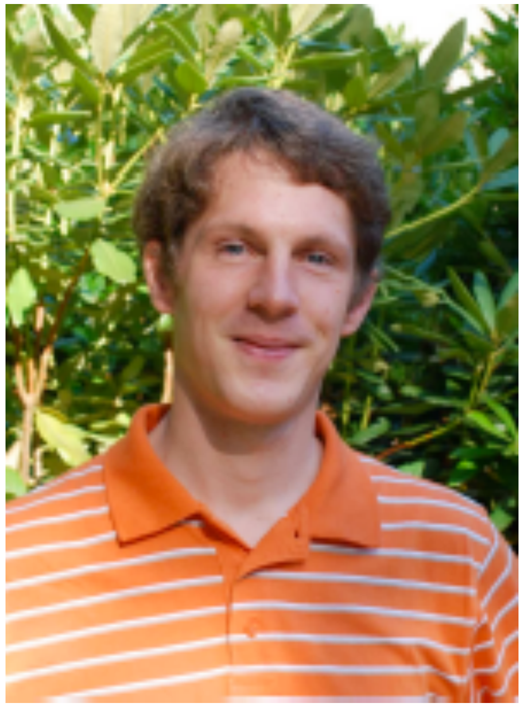

Classical linear consensus theory

Linear consensus algorithms are linear time-varying systems

$$x(t+1) = A(t)x(t), \qquad x(t) \in \mathbb{R}^n$$

where for each t, A(t) is row stochastic, i.e.

A is nonnegative: $a_{ij} \geq 0$

each row sums to one: $A(t)1 = 1$

Uniform convergence to $\alpha 1$ ("consensus: $x_i = x_j$ ")
is proven under uniform connectivity / irreducibility
(Tsitsiklis, Jadbabaie et al., Moreau, ...)

Convergence analysis and Lyapunov functions

Tsitsiklis (1986) observed that

$$V(x) = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i$$

is non increasing along the flow.

Uniform convergence is established by showing the strict decay of over a finite horizon.

It is known that no common quadratic Lyapunov exists in general. (See *Olshevsky & Tsitsiklis 08* for a discussion)

Birkhoff Theorem

Let K a closed solid cone in X a Banach space, with partial ordering $\preceq$.

A is *positive* if A maps $\mathring{K}$ to $\mathring{K}$

A is *monotone* if $x \preceq y \Rightarrow Ax \preceq Ay$

Theorem (G. Birkhoff, 1957):

Positive linear monotone mappings contract the Hilbert metric in $\mathring{K}$.

The contraction coefficient is $\tanh \frac{1}{4}\Delta(A)$

Note: Perron-Frobenius follows from contraction mapping theorem

Birkhoff Theorem and Hilbert metric in the positive orthant

$X = \mathbb{R}^n$ $\qquad K = \{(x_1, \ldots, x_n) : x_i \geq 0, 1 \leq i \leq n\}$

The Hilbert metric is

$$d(x,y) = \log \frac{\max(x_i/y_i)}{\min(x_i/y_i)}$$

It is a projective metric: $d(\lambda x, \mu y) = d(x,y) \qquad \lambda > 0, \ \ \mu > 0$

For A>0, the diameter is

$$\Delta(A) = \max\{\log(\frac{a_{ij}a_{pq}}{a_{iq}a_{pj}}) : 1 \leq i, j, p, q \leq n\}$$

Birkhoff Theorem and Tsitsiklis Lyapunov function

Consequence of Birkhoff result: for nonnegative linear maps that satisfy $A(t)1 = 1$, the Lyapunov function

$$d(x, 1) = max \log(x_i) - \min \log(x_i)$$

is non-increasing along solutions.

The Hilbert distance to consensus is equivalent to Tsitsiklis Lyapunov function in log coordinates.

(and captures the invariance property $d(\lambda x, \mu y) = d(x, y)$).

Remark: both are measures of $\mathrm{co}\{x_1, \ldots, x_n\}$
(Moreau's Lyapunov function).

Hilbert metric in an arbitrary cone

$$M(x, y) = \inf\{\lambda : x - \lambda y \preceq 0\}$$

$$m(x, y) = \sup\{\lambda : x - \lambda y \succeq 0\}$$

$$d(x, y) = \log\{M(x, y)/m(x, y)\}$$

Closely related metric: Thompson metric

$$d_T(x, y) = \log \max\{M(x, y), m^{-1}(x, y)\}$$

Hilbert metric in the SDP cone

$$K = \{X \in \mathbb{R}^{n\times n} \mid X = X^T \succeq 0\}$$

$$M(X,Y) = \inf\{\lambda : X - \lambda Y \preceq 0\} = \max_{\|v\|=1}\left(\frac{v^T X v}{v^T Y v}\right)$$

$$m(X,Y) = \sup\{\lambda : X - \lambda Y \succeq 0\} = \min_{\|v\|=1}\left(\frac{v^T X v}{v^T Y v}\right)$$

$$d(X,Y) = \log\left(\frac{\lambda_{\max}(Y^{-\frac{1}{2}}XY^{-\frac{1}{2}})}{\lambda_{\min}(Y^{-\frac{1}{2}}XY^{-\frac{1}{2}})}\right)$$

Closely related metric

$$d_{\mathrm{Riem}}(X,Y) = \| \log(Y^{-\frac{1}{2}}XY^{-\frac{1}{2}}) \|_F$$

Generalizations of classical consensus theory

I. Linearity is not essential, only homogeneity
(*Recent work by Gaubert et al. on generalizations of Perron-Frobenius*)

II. Consensus theory generalizes to any cone,
e.g. the cone of positive semidefinite matrices.

How to define a consensus iteration over the SDP cone ?
What for?

Non-commutative consensus theory

Stochastic maps in non-commutative spaces find applications in

I. Control and estimation of open quantum systems

II. Non-commutative symbolic coding

Stochastic maps: the usual (commutative) case

Probability space:

$$\mathcal{P} = \{p \in \mathbb{R}^n \ \mid p_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^{n} p_i = 1\}$$

Stochastic operators map probabilities to probabilities

$$A: \ A1 = 1, \ \ a_{ij} \geq 0, 1 \leq i,j \leq n$$

Consensus theory vs existence of a stationary distribution
vs graph theoretic interpretation of irreducibility:
see *Jadbabaie et al.*

Stochastic maps: the quantum (non-commutative) case

Probabilities are described by density matrices $\rho = \sum_i p_i v_i v_i^\dagger$

$$\mathcal{P} = \{\rho \in \mathbb{C}^{n\times n} \mid \rho = \rho^T \succeq 0,\ \mathrm{trace}(\rho) = 1\}$$

Completely positive maps ("quantum channels") map density matrices to density matrices. They are of the form

$$\Phi(\rho) = \sum_i L_i \rho L_i^\dagger, \quad \sum_i L_i^\dagger L_i = I$$

The dual map $\Psi(\rho) = \sum_i L_i^\dagger \rho L_i$ satisfies $\Psi(I) = I$

A non-commutative consensus problem

Repeated interactions of a quantum system give rise to the system

$$\rho(t+1) = \Phi(t)\rho(t)$$

$$\Phi(t)(\rho) = \sum_i L_i(t)\rho L_i^\dagger(t), \; \sum_i L_i^\dagger L_i = I$$

Birkhoff theorem: the Lyapunov function

$$V(\rho) = d(\rho, I) = \log \frac{\lambda_{\max}(\rho)}{\lambda_{min}(\rho)}$$

is non-increasing along the iterates of the dual system.

Convergence to a stationary density matrix upon irreducibility conditions.

Conclusions

Conic geometries are adapted to consensus theory ...
Quadratic Lyapunov functions are'nt ...

Tsitsiklis Lyapunov function is a measure of contraction of the Hilbert metric.

Birkhoff theorem (positive monotone operators contract the Hilbert metric) applies to more general cones, e.g. the SDP cone.

Opens the way to a consensus theory in noncommutative spaces, with a number of possible applications.

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How to bridge the gap between contraction measures and the i/o approach to consensus ?