Adaptive control and synchronization of complex networks

Alexander FRADKOV,

Inst.Probl. Mech. Eng., St.Petersburg, Russia

Jointly with Ibragim JUNUSSOV

St.Petersburg State University (SPbSU), Russia

fradkov@mail.ru, www.ipme.ru/ipme/labs/ccs/

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- **Synchronization of dynamical systems: history and definitions**
- **Decentralized output feedback synchronization for networks of Lurie systems**
  - **Problem statement**
  - **Case of identical nodes**
  - **Case of nonidentical nodes**
- **Diffusive coupling and adaptive consensus**
- **Adaptive observer-based synchronization under communication constraints**
- **Conclusions**

NETWORK SYSTEMS

Fast growth of interest during last decade:

Search for papers with “complex networks” in titles

IEEE Xplore (www.ieee.org)

2000-2001гг – 44 papers; 2005-2006гг – 101 papers

APS journals (www.aps.org)

2000-2001гг – 4 papers; 2005-2006гг – 96 papers

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Paper by R. Albert and A.-L. Barabasi, “Statistical mechanics of complex networks,”

Rev. Mod. Phys., vol. 74, pp. 47–97, 2002.

had more than 3000 citations in 2007.

NETWORK SYSTEMS (Murray, R.M., K.J. Astrom, S.P. Boyd, R.W. Brockett, and G. Stein. Future directions in control … *CS-M Apr 03* 20-33)

Problems:

- **Internet congestion control**
- **Control of energy, manufacturing, transportation networks**
- **Coordination of mobile robots, UAVs, UUVs**
- **Control of bio- and ecosystems, monitoring of global changes**
- **Quantum computation networks**

Challenges:

- **Control in geterogeneous nets (communication, computation, transportation, finance, …**
- **Reliability**
- **Asynchronous, multi-agent, distributed data processing**

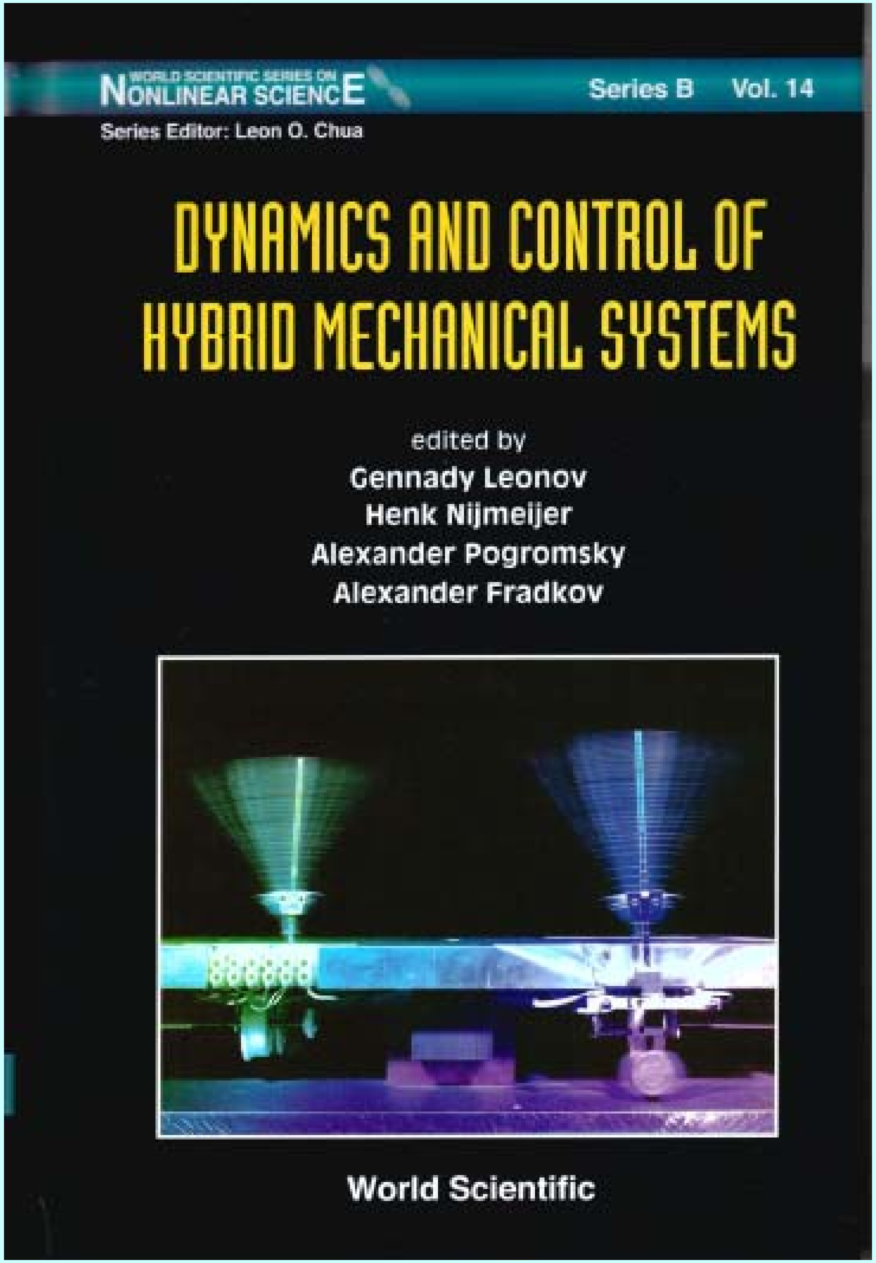
I. Synchronization of dynamical systems

1. HISTORY OF DEFINITIONS OF SYNCHRONIZATION

- **Blekhman I.I.** Synchronization of dynamical systems. Moscow: Nauka Publishers, 1971, 894 p.
  - Frequency synchronization of processes $x_1(t), \ldots, x_k(t)$ as coincidence of average frequencies of rotations or oscillations;
  - General idea of synchronization as coincidence of some functionals.
- **Gurtovnik A.S., Neimark Yu.I.** Appl. Math. Mech. 1974, No 5.
  - Convergence of $X(t) = \{x_1(t), \ldots, x_k(t)\}$ to asymptotically stable invariant torus of dimension $n - m$ (m – degree of synchronization).
- **Afraimovich V.S., Verichev N.N., Rabinovich M.I.** Radiophys. & Quantium Electronics, 1986.
  - Existence of a homeomorphism $g : \pi_1(A_c) \to \pi_2(A_c)$, such that $g(\pi_1(x_1(t))) \to \pi_2(x_2(t + \alpha(t)))$.

See also **Fujisaka H. and Yamada T. (1983)**.

I. Synchronization of dynamical systems



Dedicated to Ilya Izrailevich Blekhman on the occasion of his 80th birthday

Synchronization of dynamical systems (ctd)

- **Babloyantz A., Destezhe A. (1986), Anischenko V.S. et al. (1990–91), Landa P.S., Rosenblum M.G. (1992–93)**
  - Inequality $d < d_1 + d_2 + \cdots + d_k$,

$$\begin{aligned} \text{where } \; d \;\; &- \text{fractal dimension of } \; \{x_1, \ldots, x_k\}, \\ d_i \;\; &- \text{fractal dimension of } \; \{x_i\}. \end{aligned}$$

$$m = \sum_{i=1}^{k} d_i - d \quad - \text{degree of synchronization.}$$

- **The 1990s:**

Generalized synchronization (**Abarbanel H.D.I., Rulkov N.F., Suschik M., 1995**);

Phase synchronization (**Rosenblum M.G., Pikovsky A.S. Kurths J., 1996**);

Lag synchronization (**Rosenblum M.G., Pikovsky A.S. Kurths J., 1997**);

Master-slave synchronization (**Pecora L.M. and Carroll T.L., 1990**)

Synchronization of dynamical systems (ctd)

2. UNIFIED DEFINITIONS OF SYNCHRONIZATION

- **Blekhman I.I., Fradkov A.L., Nijmeijer H., Pogromsky A. Yu.** Systems & Control Lett., v. 31, 1997. (Special issue: Control of Chaos and Synchronization).
- **Brown R., Kocarev L.** Chaos, v. 10, No 2, 2000.
- **Boccaletti S., Pecora L., Pelaez A.** Phys. Rev. E, v. 63, 2001.

GIVEN:

– *Processes* $x_1(t)$, $x_1(t)$, $\ldots$, $x_k(t)$, $x_i(\cdot) \in \mathcal{X}$
– *Characteristic (index)* – non-anticipating family of mappings
$C_t : \mathcal{X} \to \mathcal{C}$ ($\mathcal{C}$ – metric space)
– *Comparison functions* $F_i : \mathcal{C} \to \Re^m$, $i = 1, \ldots, k$

DEFINITION. Processes $x_1(t)$, $\ldots$, $x_k(t)$ are called

A) **synchronized** w.r.t. index C_t and comparison functions F_i, $i = 1, \ldots, k$, if there exist time shifts τ_i, $i = 1, \ldots, k$:

$$F_1\left(C_{t+\tau_1}[x_1]\right) = F_2\left(C_{t+\tau_2}[x_2]\right) = \ldots = F_k\left(C_{t+\tau_k}[x_k]\right) \qquad \forall t \geq 0$$

Definitions of synchronization (ctd)

B) ε–**synchronized** w.r.t. index C_t and comparison functions F_i, $i = 1, \dots, k$, if there exist time shifts τ_i, $i = 1, \dots, k$:

$$\left\| F_i\left(C_{t+\tau_i}[x_i]\right) - F_j\left(C_{t+\tau_j}[x_j]\right)\right\| \le \varepsilon \qquad \forall i, j; \quad t \ge 0$$

C) **asymptotically synchronized** w.r.t. index C_t and comparison functions F_i, $i = 1, \dots, k$, if there exist time shifts τ_i, $i = 1, \dots, k$:

$$\lim_{t\to\infty} \left\| F_i\left(C_{t+\tau_i}[x_i]\right) - F_j\left(C_{t+\tau_j}[x_j]\right)\right\| = 0$$

D) **synchronized in the average** w.r.t. index C_t and comparison functions F_i, $i = 1, \dots, k$, if there exist time shifts τ_i, $i = 1, \dots, k$:

$$\langle Q_t \rangle < \varepsilon$$

$$Q_t = \sum_{i,j=1}^{k} \left\| F_i\left(C_{t+\tau_i}[x_i]\right) - F_j\left(C_{t+\tau_j}[x_j]\right)\right\|^2 -$$

– measure of synchronization.

I.I.Blekhman, A.L.Fradkov. On general definitions of synchronization. In: Selected topics in vibrational mechanics, Ed. I.I.Blekhman, Singapore: World Scientific, 2004, pp. 179-188.

Definitions of synchronization. Examples

1. Frequency synchronization:

$$C_t[x] = \langle \dot{x} \rangle_t = \omega_t \quad -\text{average frequency of } x(s) \text{ on } [0,t].$$

2. Phase synchronization:

A) $C_t[x] = \varphi_t = 2\pi \frac{t-t_n}{t_{n+1}-t_n} + 2\pi n$, $t_n \le t < t_{n+1}$, t_n – time of nth crossing of the Poincaré section.

$$k=2: \quad F_1(\varphi_t) = F_2(\varphi_t) = \varphi_t \qquad -\text{inphase synchronization}$$

$$F_1(\varphi_t) = \varphi_t, \quad F_2(\varphi_t) = \varphi_t + \pi \quad -\text{antiphase synchronization}$$

B) $C_n[x] = t_n$.

3. Complete coordinate (identical) synchronization:

$$C_t[x] = x(t), \quad F_i(x) = x, \quad i = 1, \ldots, k.$$

4. Generalized coordinate (partial) synchronization:

$$C_t[x] = x(t) \qquad x(t) \in \Re^n,$$

$$F_1(x) = G_1(x), \;\; F_2(x) = G_2(x), \;\; G_i : \Re^n \to \Re^m.$$

Remark. Asymptotic coordinate synchronization = coordination=convergence=consensus

Partial coordinate synchronization=output synchronization

3. DEFINITION OF ADAPTIVE SYNCHRONIZATION

(A.L.Fradkov. Nonlinear adaptive control: regulation-tracking-oscillations. Proc. 1st IFAC Workshop "New trends in design of control systems", 1994, 426-431.)

Given equations of controlled systems

$$\dot{x}_i = \overline{F}_i(x_1, \ldots x_r, u, t, \xi), \quad i=1\ldots r \qquad (2.3)$$

where $\xi \in \Xi$ is vector of unknown parameters, find the equation of the control algorithm

$$u = U(x_1, \ldots x_r, \theta, t)) \qquad (2.4)$$

and adaptation algorithm

$$\dot{\theta} = \Theta(x_1, \ldots x_r, \theta, t) \qquad (2.5)$$

ensuring the control goal

$$\| x_i(t) - \overline{x}_i(t) \| \leq \Delta \text{ for } t > t_* \qquad (2.6)$$

Adaptive synchronization of networks

B. Gazelles, B. Boudjema, N.P. Chau. Resynchronisation of globally coupled chaotic oscillators using adaptive control. Phys. Lett. A 210 (1996), 95-100 *(See also Physica* D 103: pp. 452-465, 1997.)

Z. Li and G. Chen, "Robust adaptive synchronization of uncertain dynamical networks," Phys. Lett. A, v.324, pp.166-178, 2004.

J.Zhou, Jun-an Lu, J.Lu. Adaptive Synchronization of an Uncertain Complex Dynamical Network. IEEE Tr. AC-51, (4), 2006.

Jing Yao, David J. Hill, Zhi-Hong Guan, and Hua O. Wang. Synchronization of Complex Dynamical Networks with Switching Topology via Adaptive Control. Proc. 45th IEEE CDC, San Diego, CA, 2006.

Z. Yang, Z. Liu, Z.Chen, Z. Yuan. Adaptive Synchronization of an Uncertain Complex Delayed Dynamical Networks. Int. J. Nonlin. Sci. V. 3 (2007) N2.

T. Liu, G. Dimirovski, J. Zhao. Controlled Synchronization of Complex Dynamical Networks with Nonlinear Nodes and Couplings. 3rd IEEE Multiconf. Systems and Control, St.Petersburg, July 8-10, 2009.

P.De Lellis, M. di Bernardo, F.Garofalo. Novel decentralized adaptive strategies for the synchronization of complex networks. Automatica 45 (2009) 1312-1318.

Typical dynamical network model

(J.Zhou, Jun-an Lu, J.Lu. Adaptive Synchronization of an Uncertain Complex Dynamical Network. IEEE Trans. Autom. Control, (4), 2006.)

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) + \mathbf{u}_i \qquad (1)$$

where $1 \leq i \leq N$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbf{R}^n$ is the state vector of the ith node, $\mathbf{f} : \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$ is a smooth nonlinear vector field, node dynamics is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, $\mathbf{h}_i : \Omega \times \cdots \times \Omega \rightarrow \mathbf{R}^n$ are unknown nonlinear smooth diffusive coupling functions, $\mathbf{u}_i \in \mathbf{R}^n$ are the control inputs, and the coupling-control terms satisfy $\mathbf{h}_i(\mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}) + \mathbf{u}_i = \mathbf{0}$, where $\mathbf{s}$ is a synchronous solution of the node system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

Typical adaptive control algorithm

(J.Zhou, Jun-an Lu, J.Lu. Adaptive Synchronization of an Uncertain Complex Dynamical Network. IEEE Trans. Autom. Control, 2006 (4).)

Hypothesis 1: H1) Assume that there exists a nonnegative constant α satisfying $\|D\mathbf{f}(\mathbf{s}, t)\|_2 = \|\mathbf{A}(t)\|_2 \leq \alpha$, where $\mathbf{A}(t)$ is the Jacobian of $\mathbf{f}(\mathbf{s}, t)$.

Hypothesis 2: H2) Suppose that there exist nonnegative constants γ_{ij} $(1 \leq i, j \leq N)$ satisfying $\|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s})\|_2 \leq \sum_{j=1}^{N} \gamma_{ij} \|\mathbf{e}_j\|_2$ for $1 \leq i \leq N$.

$$\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t), \qquad 1 \leq i \leq N.$$

$$\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) = \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) - \mathbf{h}_i(\mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}).$$

Typical adaptive control algorithm (ctd)

(J.Zhou, Jun-an Lu, J.Lu. Adaptive Synchronization of an Uncertain Complex Dynamical Network. IEEE Trans. Autom. Control, (4), 2006.)

Theorem 1: Suppose that H1 and H2 hold. Then, the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (1) is locally asymptotically stable under the adaptive controllers

$$\mathbf{u}_i = -d_i \mathbf{e}_i, \qquad 1 \le i \le N \tag{7}$$

and updating laws

$$\dot{d}_i = k_i \mathbf{e}_i^T \mathbf{e}_i = k_i \|\mathbf{e}_i\|_2^2, \qquad 1 \le i \le N \tag{8}$$

Challenge:
To design decentralized adaptive output feedback control ensuring synchronization under conditions of uncertainty and incomplete control

II. Decentralized output feedback adaptive synchronization for networks of Lurie systems

I.Junussov&A.Fradkov. Decentralized adaptive output feedback synchronization of interconnected Lurie systems. Autom.Rem.Control. 2009, 7.

A.Fradkov&I.Junussov, R.Ortega. IEEE MSC'09, St.Petersburg, July, 2009.

A.Fradkov&I.Junussov. IPACS Physcon09, Catania, 2009.

Problem statement I: System description

$$\dot{\overline{x}} = A_L \overline{x} + B_L(\overline{u} + \psi_0(\overline{y})), \quad \overline{y} = C^{\mathrm{T}} \overline{x}, \tag{1}$$

where $\overline{x} \in \mathbb{R}^n$ – state, $\overline{y} \in \mathbb{R}^l$ – measurment, $\overline{u}(t) \in \mathbb{R}^1$ is control that specified in advance, $\psi_0 \colon \mathbb{R}^l \to \mathbb{R}^1$ – internal nonlinearity. Assume A_L, B_L, C and $\psi_0(\cdot)$ **are known and do not depend** of unknown parameters $\xi \in \Xi$, where Ξ is known set. Consider a network S of d interconnected subsystems S_i, $i = 1, \ldots, d, d \in \mathbb{N}$. Subsystem S_i :

$$\dot{x}_i = A_i x_i + B_i u_i + B_L \psi_0(y_i) + \sum_{j=1}^{d} \alpha_{ij} \varphi_{ij}(x_i - x_j), \tag{2}$$

$$y_i = C^{\mathrm{T}} x_i, \qquad i = 1, \ldots, d,$$

where $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^1, \alpha_{ij} \in \mathbb{R}^1, y_i \in \mathbb{R}^l$. Functions $\varphi_{ij}(\cdot)$, $i = 1, \ldots, d$, $j = 1, \ldots, d$, describe interconnections between subsystems. . Let matrices A_i, B_i and functions $\varphi_{ij}(\cdot)$, $i = 1, \ldots, d, j = 1, \ldots, d$, depend on the vector of unknown parameters $\xi \in \Xi$.

Problem statement II: Control goal

Let the control goal be specified as convergence of all trajectories of subsystems and the leader:

$$\lim_{t\to+\infty} (x_i(t) - \overline{x}(t)) = 0, \qquad i = 1, \ldots, d. \qquad (3)$$

The adaptive synchronization problem is to find a decentralized controller $u_i = \mathcal{U}_i(y_i, \overline{u}, t)$ ensuring the goal (3) for all values of unknown initial conditions $x_i(0), \bar{x}(0)$ and all values of unknown plant parameters $\xi \in \Xi$.

Remark. Control goal (3) corresponds to **coordinate synchronization** or **convergence** or **consensus:**

$$|x_i(t) - x_j(t)| \to 0 \quad \text{as} \quad t \to \infty,\ i, j = 1, \ldots d.$$

Decentralized output feedback synchronization for identical nodes: $A_i = A$, $B_i = B$

Adaptive controller:

$$u_i = \theta_i^{\mathrm{T}}(t)\,[y_i - \overline{y}] + \overline{u},$$

$$\dot{\theta}_i = -g^{\mathrm{T}}\tilde{y}_i \Gamma_i \tilde{y}_i, \qquad i = 1, \ldots, d,$$

If the functions ψ_0, φ_{ij} are Lipschitz continuous, and $g^T C^T[(s+L)I-A]^{-1}B$ – hyper minimum-phase and

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}|$$

is sufficiently small, then the adaptive controller ensures the goal and $\theta_i(t)$ are bounded.

(Adaptive synchronization of nonlinear dynamical networks.
I.Junussov, A.Fradkov, Autom. Remote Control, 2009, N7)

Passification theorem

(A.L.Fradkov: Aut.Rem.Contr.,1974(12), Sib.Math.J. 1976, Eur. J.Contr. 2003; Andrievsky, Fradkov, Aut.Rem.Contr., 2006 (11))

Let $W(s)=C^T(sI-A)^{-1}B$ – transfer matrix of a linear system. The following statements are equivalent:

A) There exist matrix $P=P^T>0$ and row vector K, such that

$PA_K+A_K^T P<0$, $PB=(CG^T)$, $A_K=A+BKC^T$

B) Matrix GW(s) is hyper-minimum-phase (det[GW(s)] has Hurwitz numerator, $GC^TB=(GC^TB)^T>0$)

C) There exists a K, such that feedback u=Ky+v renders the system strictly passive «from input v to output Gy»

Case of nonidentical nodes.

(A.Fradkov, I.Junussov, Physcon-2009.)

Adaptive controller

Denote $\sigma_i(t) = \mathrm{col}(y_i(t), \overline{u}(t))$. Let the main loop of the adaptive system be specified as set of linear tunable local control laws:

$$u_i(t) = \tau_i(t)^{\mathrm{T}}\sigma_i(t), \quad i = 1, \ldots, d, \tag{4}$$

where $\tau_i(t) \in \mathbb{R}^{l+1}, i = 1, \ldots, d$ are tunable parameters. By applying speed-gradient method we can derive following adaptation law:

$$\dot{\tau}_i = -g^{\mathrm{T}}(y_i - \overline{y})\Gamma_i\sigma_i(t), i = 1, \ldots, d, \tag{5}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ – $(l+1) \times (l+1)$ matrices, $g \in \mathbb{R}^l$.

Assumptions

A1) The functions $\varphi_{ij}(\cdot)$,$i = 1, \ldots, d$, $j = 1, \ldots, d$ are globally Lipschitz:

$$\|\varphi_{ij}(x) - \varphi_{ij}(x')\| \le L_{ij}\|x - x'\|, \quad L_{ij} > 0,$$
$$i = 1, \ldots, d, \quad j = 1, \ldots, d.$$

A2)(matching conditions) For each $\xi \in \Xi$, $i = 1, \ldots, d$ there exist vectors $\nu_i = \nu_i(\xi) \in \mathbb{R}^l$ and numbers $\theta_i = \theta_i(\xi) > 0$ such that

$$A_L = A_i + B_i \nu_i^{\mathrm{T}} C^{\mathrm{T}}, B_L = \theta_i B_i. \tag{6}$$

Notations

Consider real matrices $H = H^{\mathrm{T}} > 0, g$ of size $n \times n, l \times 1$ correspondingly and a number $\rho > 0$ such that:

$$HA_L + A_L^{\mathrm{T}} H < -\rho H, \quad HB_L = Cg. \tag{7}$$

Denote $\lambda_* = \lambda_{max}(H)/\lambda_{min}(H)$ condition number of matrix H, where $\lambda_{max}(H), \lambda_{min}(H)$ are maximum and minimum eigenvalues of matrix H.
Denote $\chi(s) = C^{\mathrm{T}}(sI_n - A_L)^{-1}B_L$. For the case when matrix A_L is Hurwitz introduce notation ρ_* for degree of stability of function's $g^{\mathrm{T}}\chi(s)$ denominator, i.e. $\rho_* = \min_{k=1,\ldots,n} |\mathrm{Re}\,\lambda_k(A_L)|$ where $\lambda_k(A_L)$ are eigenvalues of A_L.

Definition

Let $G \in \mathbb{R}^l$. Function $f : \mathbb{R}^l \to \mathbb{R}^1$ is called G-monotonically decreasing if inequality $(x - y)^{\mathrm{T}} G\,(f(x) - f(y)) \leq 0$ holds for all $x, y \in \mathbb{R}^l$.

Synchronization conditions

Theorem (1)

Let $B \neq 0$, matrix A_L is Hurwitz and for some $g \in \mathbb{R}^l$ following frequency domain conditions hold:

$$\mathrm{Re}\, g^{\mathrm{T}}\chi(i\omega) > 0, \qquad \lim_{\omega\to\infty} \omega^2\, \mathrm{Re}\, g^{\mathrm{T}}\chi(i\omega) > 0 \tag{8}$$

for all $\omega \in \mathbb{R}^1$.

. Let for all $\xi \in \Xi$ assumptions A1, A2 hold, function $\psi_0(\cdot)$ be g-monotonically decreasing, and following inequalities hold

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| < \gamma \quad i = 1, \ldots, d \tag{9}$$

where $\gamma = \rho_* / (2\lambda_*)$, λ_* is condition number of matrix H.
Then for all $\xi \in \Xi, i = 1, \ldots, d$ adaptive controller (4),(5) ensures achievement of the goal (3) and boundedness of functions $\theta_i(t)$ on $[0, \infty)$ for all solutions of the closed-loop system (1), (2), (4), (5).

Remark 1. Evaluation of the value of γ

The value of γ can be evaluated by solving LMI

$$HA_L + A_L^{\mathrm{T}} H < -\rho H, \quad HB_L = Cg,$$

by means of one of existing software package $(0 < \rho < \rho_*)$.

Remark 2. Weighted in-degree of graph node

The inequality (9) from Theorem 1:

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| < \gamma \quad i = 1, \ldots, d$$

can be interpreted as follows:

Weighted in-degree of the each node of connections graph must be less than γ.

Example: Network of Chua circuits

Let the leading subsystem be described by the equation

$$\dot{\overline{x}} = A_L \overline{x} + B_L(\overline{u} + \psi_0(\overline{y})), \quad \overline{y} = C^{\mathrm{T}} \overline{x},$$

where $\overline{x} \in \mathbb{R}^3$ is state vector of the system, $\overline{y} \in \mathbb{R}^1$ is output available for measurement, $\overline{u}$ is scalar control variable, $\psi_0(\overline{y}) = pv(\overline{y})/b$, where $v(x) = -0.5(m_0 - m_1)(|x+1| - |x-1| - 2x)$. Further, let

$$A_L = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix},$$

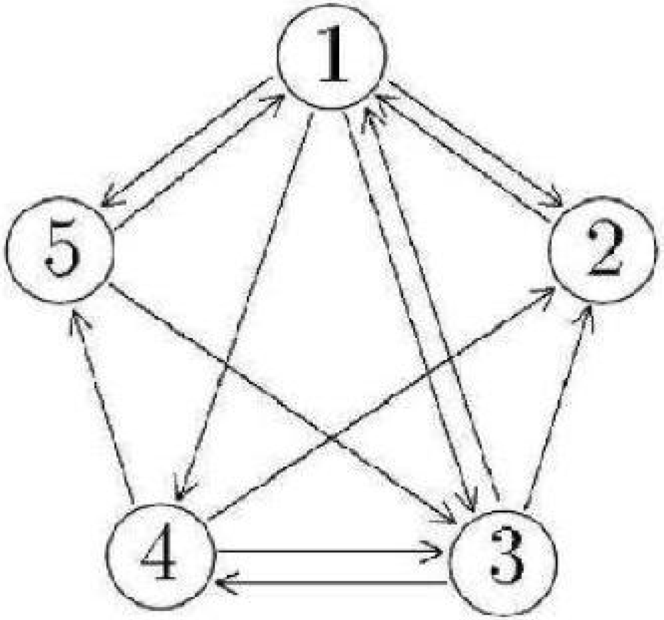
$B_L = \mathrm{col}(b, 0, 0), C = \mathrm{col}(1, 0, 0)$. Let the leader be controlled as follows:

$$\overline{u}(t) = \frac{1}{b}\left[(-(1 + m_0)p + 1)\overline{x}_1(t) + p\overline{x}_2(t)\right].$$

Such control ensures chaotic behavior of the leader for $m_0 = -8/7, m_1 = -5/7, p = 15.6, q = 30, b = 1$ and $g = 1$.

Example: Network of Chua circuits (ctd)

Denote $\varphi_{ij} = \varphi_{ij}(x_i - x_j)$, $i = 1, \ldots, 5$, $j = 1, \ldots, 5$. Let $\varphi_{14}, \varphi_{25}, \varphi_{32}, \varphi_{42}, \varphi_{45}, \varphi_{52}, \varphi_{53}$, be equal to $(0, 0, 0)^{\mathrm{T}}$. Further, let



$$\varphi_{12} = (\sin(x_{11} - x_{21}), 0, 0)^{\mathrm{T}},$$
$$\varphi_{13} = (0, x_{12} - x_{32}, 0)^{\mathrm{T}},$$
$$\varphi_{15} = (0, 0, \sin(x_{13} - x_{53}))^{\mathrm{T}},$$
$$\varphi_{21} = (x_{21} - x_{11}, 0, x_{23} - x_{13})^{\mathrm{T}},$$
$$\varphi_{23} = (0, \sin(x_{22} - x_{32}), 0)^{\mathrm{T}},$$
$$\varphi_{24} = (0, x_{22} - x_{42}, 0)^{\mathrm{T}},$$
$$\varphi_{31} = (\sin(x_{31} - x_{11}), 0, 0)^{\mathrm{T}},$$
$$\varphi_{34} = (\sin(x_{31} - x_{41}), 0, 0)^{\mathrm{T}},$$
$$\varphi_{35} = (x_{31} - x_{51}, x_{32} - x_{52}, x_{33} - x_{53})^{\mathrm{T}},$$
$$\varphi_{41} = (0, \sin(x_{42} - x_{12}), 0)^{\mathrm{T}},$$
$$\varphi_{43} = (\sin(x_{41} - x_{31}), 0, 0)^{\mathrm{T}},$$
$$\varphi_{51} = (x_{51} - x_{11}, 0, x_{53} - x_{13})^{\mathrm{T}},$$
$$\varphi_{54} = (0, x_{52} - x_{42}, 0)^{\mathrm{T}}.$$

Example: Network of Chua circuits (ctd)

Let subsystem S_i for $i = 1, \ldots, 5$ be described by (2) with $u_i, \alpha_{ij} \in \mathbb{R}^1$. By choosing $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = (3, 1, 4, 1, 5), \theta_i = 1/i, i = 1, \ldots, 5$ and using (6) we obtain matrices A_i, B_i for $i = 1, \ldots, 5$, which are not equal, i.e. nodes are nonidentical.

It is easy to show that frequency-domain conditions (8) hold, $\psi_0(\cdot)$ is g-monotonically decreasing, Lipschitz constants of all φ_{ij} are equal to 1.

Example: Simulation for 40 seconds

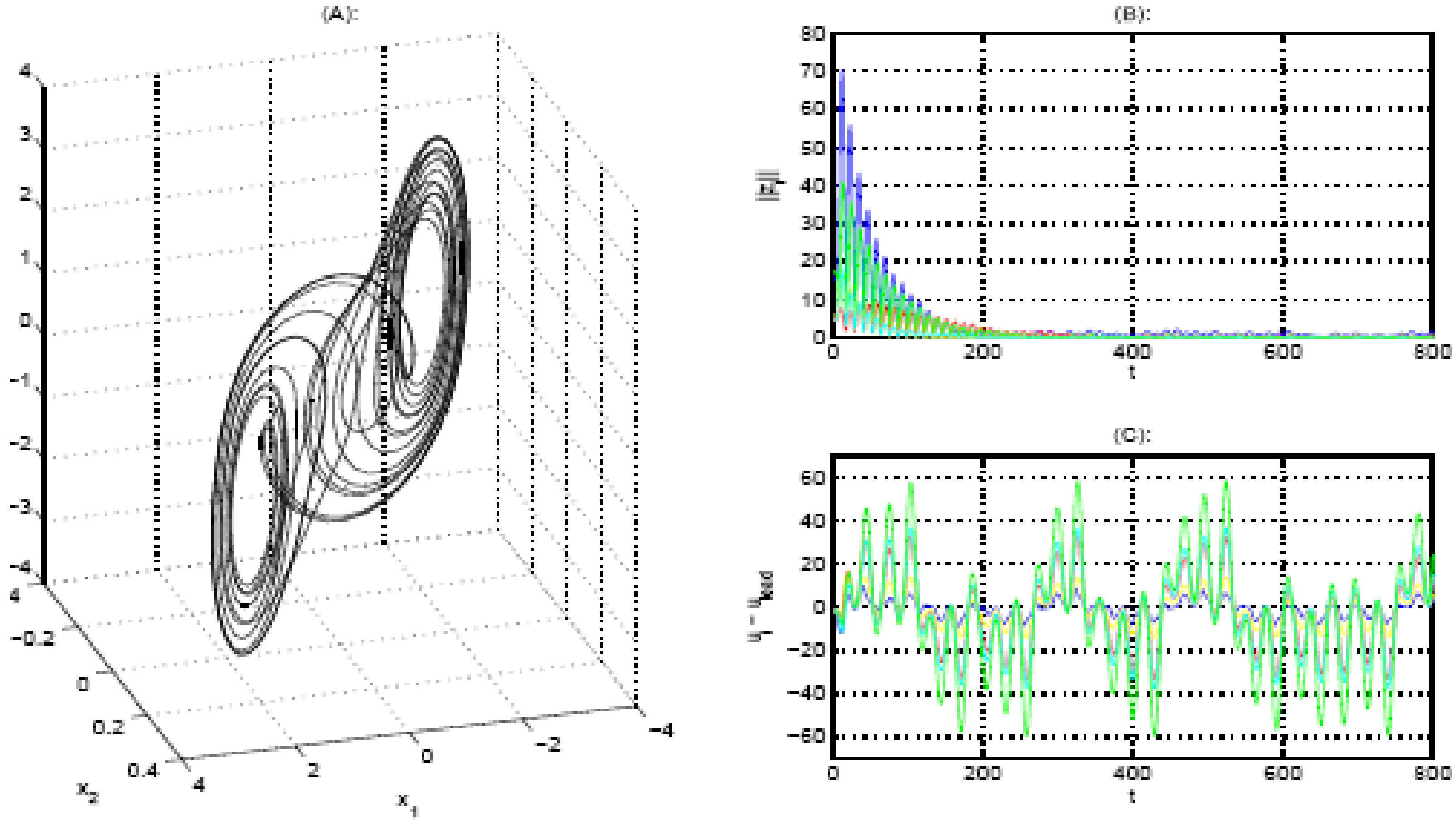


Figure: (A): Phase portrait of leading subsystem, (B): $\|x_i - \overline{x}\|$, $\tilde{u}_i = u_i - \overline{u}, i = 1, \ldots, 5$.

ADAPTIVE SYNCHRONIZATION OF NETWORKS BY DIFFUSION COUPLING

I.A.Junussov, A.L Fradkov. XI International E.S. Pyatnitskiy Workshop STAB 10, June 1-4, 2010, Moscow

Controlled network:

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = C^{\mathrm{T}} x_i,$$

Diffusion coupling (Output Feedback, Consensus Protocol):

$$u_i = K \sum_{j \in N_i} (y_i - y_j) = KC^{\mathrm{T}} \sum_{j \in N_i} (x_i - x_j), \quad K \in \mathbb{R}^{1 \times l},$$

Theorem.
A1. Network graph is strongly connected and balanced.
A2. gW(s) is hyper-minimum-phase for some $g=(g_1,\dots g_n)$.
Let K=μg, where μ>0 is sufficiently large.

Then synchronization is achieved:

$$\lim_{t\to\infty}(x_i(t)-d^{-1/2}e^{At}(1_d^{\mathrm{T}}\otimes I_n)x(0))=0, i=1,\dots,d.$$

Extension 1. Network of Lurie systems.
Extension 2. Network graph has a spanning tree
Extension 3. Adaptive controller:

$$u_i=\mu_i^T \sum_{j\in N_i} (y_i - y_j)$$

$$d\mu_i/dt=\gamma g \sum_{j\in N_i} (y_i - y_j) \sum_{j\in N_i} (y_i - y_j)$$

III. ADAPTIVE SYNCHRONIZATION OF NONLINEAR NETWORKS UNDER COMMUNICATION CONSTRAINTS

Boris Andrievsky, Alexander Fradkov. X International E.S. Pyatnitskiy Workshop, June 3-6, 2008, Moscow

PLAN

1. **Introduction**
2. **Synchronization of nonlinear systems under communication constraints**
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   - **2.2. Problem statement**
   - **2.3. Coding procedure for first order coder**
   - **2.4. Bound for synchronization error**
   - **2.5. Synchronization of chaotic Chua systems**
3. **Adaptive synchronization of nonlinear systems and networks under communication constraints**

Conclusions

***G. Nair, R. Evans*, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, 2003, 585-593.**

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k,$$

$$Z = \{0, 1, \dots, \mu - 1\}, \quad R = \log_2 \mu$$

$$s_k = \gamma_k(\widehat{y}_k, \widehat{s}_{k-1}) - \textbf{coder}$$

$$u_k = \delta_k(\widehat{s}_{k-1}) - \textbf{controller}$$

ρ **- exponential stabilizability of the system:**

$$\rho^{-kr} \|x_k\|^r \to 0 \text{ as } k \to \infty \iff$$

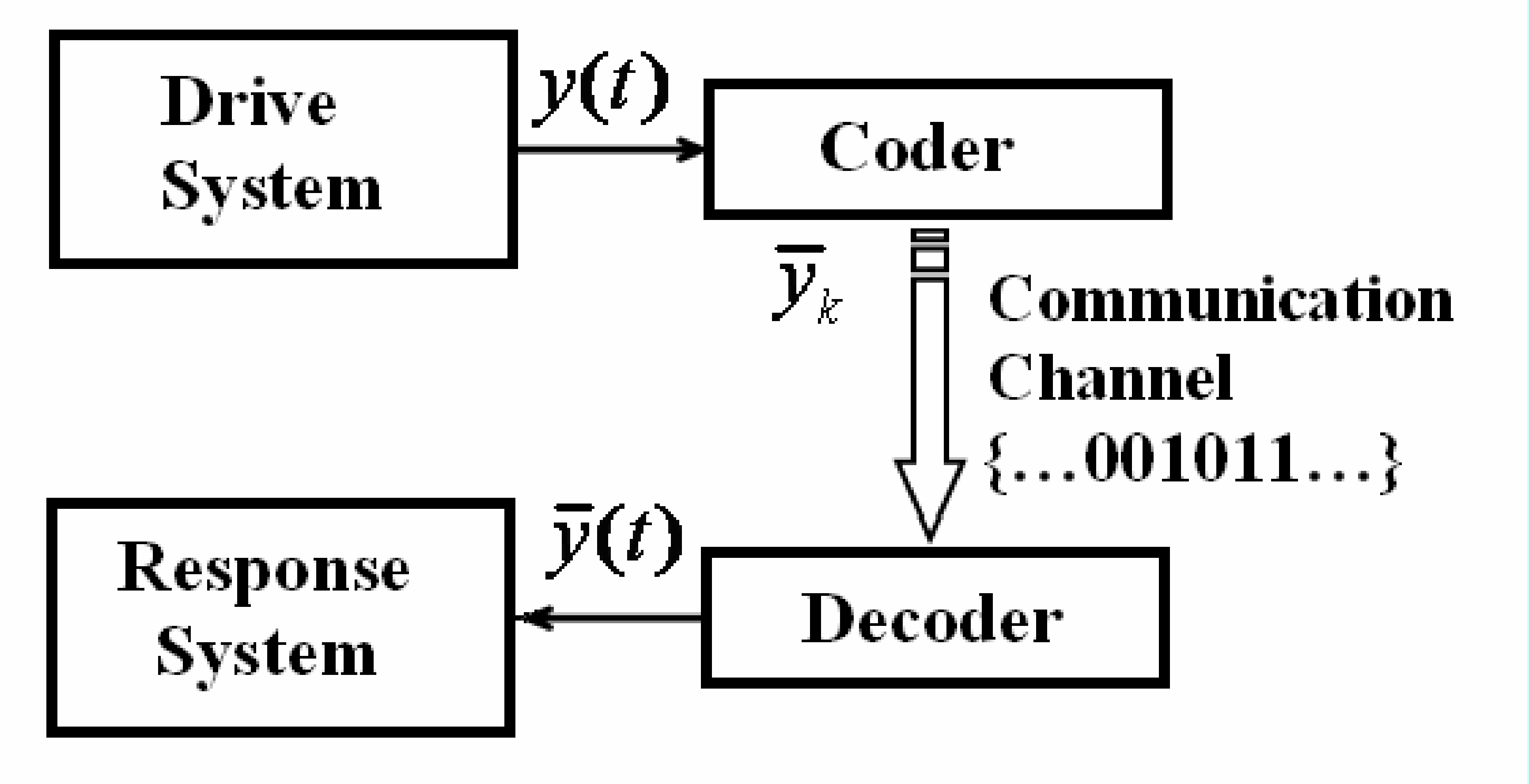
$$R > \sum_{|\eta_j| \geq \rho} \log_2 \left| \frac{\eta_j}{\rho} \right|$$

Corollary ("Data-Rate Theorem"):

INFORMATION MUST BE TRANSPORTED AS FAST AS THE SYSTEM GENERATES IT, OR ELSE INSTABILITY OCCURS

Observer-based Synchronization of Nonlinear Systems Under Communication Constraints

Drive–response synchronization of two unidirectionally coupled oscillators via a communication channel with finite capacity:



Coupled systems in the Lurie form

- **Drive system:**

$$\dot{x} = Ax + \varphi(y), \ y = Cx, \qquad (1)$$

x **-** ***n*****-dimensional vector of state variables,** ***y*** **- scalar output,** ***A*** **- (*****n*** **x** ***n*****)-matrix,** ***C*** **- (1 x** ***n*****)-matrix, φ(*****y*****) – nonlinearity. All the trajectories of (1) belong to a bounded set Ω.**

- **Response system - nonlinear observer:**

$$\dot{\hat{x}} = A\hat{x} + \varphi(y) + K(y - \hat{y}), \ \hat{y} = C\hat{x}, \qquad (2)$$

K **- observer gain vector.**

Coding procedure

Central numbers **c_k, k = 0,1,...; c_0 = 0. Deviation $\partial y_k = y_k - c_k$,**

$$q_{\nu,M}(y) = \begin{cases} \delta \cdot \langle \delta^{-1} y \rangle, & \text{if } |y| \le M, \\ M \operatorname{sign}(y), & \text{otherwise}, \end{cases} \qquad (6)$$

Signal ∂y_k transmitted over the channel to the decoder.

Numbers c_{k+1} and M_k are renewed:

$$c_{k+1} = c_k + \partial y_k, \quad c_0 = 0, \quad k = 0, 1, \ldots, \qquad (8)$$

$$M_k = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0, 1, \ldots, \qquad (9)$$

$0 < \rho \le 1$ - *decay parameter*, M_∞ - limit value of M_k.

M_0: should be large enough to capture all y_0.

Transmission error

$$\delta_y(t) = y(t) - \bar{y}(t). \qquad (4)$$

Error equation:

$$\dot{e} = A_K e + \varphi(y) - \varphi\big(y + \delta_y(t)\big) - K\delta_y(t) \qquad (5)$$

where $A_K = A - KC$. **Let** $|\delta_y(t)| \le \Delta$.

Upper bound of the limit synchronization error:

$$Q = \sup \overline{\lim_{t\to\infty}} \|e(t)\|$$

Problem: relate Q to Δ and Δ to data rate R.

Bound for synchronization error

Assumption: $|\varphi(y) - \varphi(y+\delta)| \le L_\varphi|\delta|$ for all $y=Cx$, x in Ω, Ω - a set containing all the trajectories of the drive system (1).

Choose K s.t. A_K is a Hurwitz matrix; choose $P=P^{\mathrm{T}}>0$ satisfying for some $\mu > 0$ the modified Lyapunov inequality:

$$PA_K + A_K^{\mathrm{T}}P \le -\mu P, \qquad (21)$$

Result:

$$\overline{\lim_{t\to\infty}}\|e(t)\| \le C_e^{+} r^{*} L_y/\bar{R}, \qquad (24)$$

where

$$C_e^{+} = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,\frac{L_\varphi + \|K\|}{\mu}.$$

[Fradkov, Andrievsky, Evans. Physical Review E, v.73, 066209, 2006]

Adaptive Synchronization of Nonlinear Systems Under Communication Constraints

[Fradkov A.L., Andrievsky B., Evans R.J. Proc. 1st IFAC Conf. "CHAOS 2006", **Reims, 28-30 June, 2006. pp. 279-284.**
http://www.ifac-papersonline.net

Fradkov A.L., Andrievsky B., Evans R.J. **Adaptive Observer-Based Synchronization of Chaotic Systems with First-Order Coder in Presence of Information Constraints.**
IEEE Trans. Circuits & Systems-I, 2008]

Problem statement and synchronization scheme

Nonlinear uncertain system ("transmitter", "master system"):

$$\dot{x} = Ax + \varphi_0(y) + B\sum_{i=1}^{m}\theta_i\varphi_i(y), \quad y = Cx, \qquad (1)$$

x **- transmitter state** n**-vector;** ***y*** **–** l**-vector of outputs (to be transmitted over the communication channel);**

$\theta = [\theta_1, \dots, \theta_m]^T$ **- parameters.**

Assumption: $\varphi_i(\cdot)$, A, C, B **are known;** $W(s)=C(sI-A)^{-1}B$ **- HMP.**

To achieve synchronization between two chaotic systems: ***adaptive observer*** **[*Fradkov, Nijmeijer, Markov, Int. J. Bifurc. Chaos*, 10 (12), 2000].**

Adaptive observer

- ***Tunable observer block*:**

$$\dot{\hat{x}} = A\hat{x} + \varphi_0(y) + B\sum_{i=1}^{m}\hat{\theta}_i\varphi_i(y) + L(y-\hat{y}),$$

$$\hat{y} = C\hat{x}, \qquad (2)$$

x **–observer state *n*-vector, *y* – observer output *l*-vector,**
$\hat{\theta}_i$ **– tunable parameters (*i* = 1,2,...,*m*).**

- ***Adaptation block*:**

$$\dot{\hat{\theta}}_i = -\gamma_i(y-\hat{y})\varphi_i(y), \quad i = 1,2,\ldots,m, \qquad (3)$$

$\gamma_i > 0$ **– adaptation gains.**

Regularized adaptive observer

Observer input: $\bar{y}(t) = y(t) + \delta_y(t)$

$\delta_y(t)$ – total distortion.

$$\dot{\hat{x}} = A\hat{x} + \varphi_0(\bar{y}) + B\sum_{i=1}^{m}\hat{\theta}_i\varphi_i(\bar{y}) + L(\bar{y} - \hat{y}),$$

$$\hat{y} = C\hat{x}, \qquad (5)$$

$$\dot{\hat{\theta}}_i = -\gamma_i(\bar{y} - \hat{y})\varphi_i(\bar{y}) - \alpha_i\hat{\theta}_i, \quad i = 1, 2, \ldots, m, \qquad (6)$$

α_i – regularization gains.

Analytical bounds for synchronization error

Limit synchronization error $Q = \overline{\lim_{t\to\infty}} \|x(t) - \hat{x}(t)\|.$

Problem: find bounds on the system parameters and transmission rate ensuring prespecified upper bound of the asymptotic error $Q \le \Delta_x$.

Theorem 1. Let the following assumptions hold:

A1. The observer gain matrix L is such that the transfer function

$$W_L(\lambda) = C(\lambda \mathrm{I} - A + LC)^{-1} B$$

is strictly passive, i.e. satisfies inequalities

$$\operatorname{Re} W_L(i\omega) > 0 \quad \forall \omega \geq 0,$$
$$\lim_{\omega \to \infty} \omega^2 \operatorname{Re} W_L(i\omega) > 0. \qquad (10)$$

A2. The system (1) possesses a bounded invariant set $\Omega_\theta \subset \mathbb{R}^n$ for any $\theta \in \Theta \subset \mathbb{R}^m$, where Θ is the set of possible values of uncertain parameters and $x(0) \in \Omega$.

A3. Functions $\varphi_i(y)$, $i = 0, 1, \ldots, m$ are bounded and Lipschitz continuous in the closed Δ-vicinity of Ω_θ, i.e.

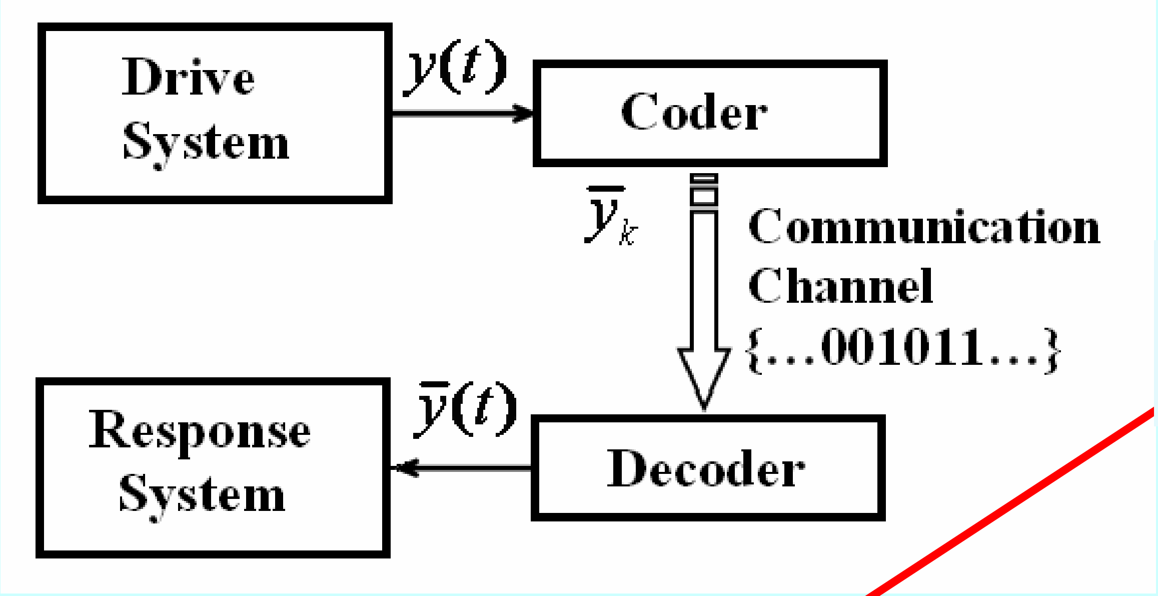
$$|\varphi_i(y)| \le L_\varphi, \quad |\varphi_i(y') - \varphi_i(y)| \le L'_\varphi$$

for some L_φ, L'_φ and for all $y = Cx$, $x \in S_\Delta(\Omega_\theta)$, where $S_\Delta(\Omega_\theta) = \{x : \exists z \in \Omega_\theta : \|x - z\| \le \Delta\}$.

Then there exist constants $C_1 > 0$, $C_2 > 0$ such that for any $\Delta > 0$ the choice of design parameters $\alpha = \Delta^2$, $\gamma = C_2/\Delta^2$ guarantees that the synchronization goal is achieved for

$$\Delta_x = C_1 \Delta,$$

Adaptive synchronization of Chua systems



Unknown parameter

Drive system

$$\begin{cases} \dot{x}_1 = p(x_2 + x_1 + f(x_1) + \theta f_1(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -qx_2, \end{cases} \qquad y(t) = x_1(t), \qquad (30)$$

$$f(z) = m_0 z + 0.5(m_1 - m_0) f_1(z), \quad f_1(z) = |z+1| - |z-1|\,.$$

Response system – adaptive observer

$$\begin{cases} \dot{\hat{x}}_1 & = p(\hat{x}_2 + \hat{x}_1 + f(\bar{y}(t)) + \hat{\theta}(t) f_1(\bar{y}(t)) + l_1 \varepsilon(t), \\ \dot{\hat{x}}_2 & = \hat{x}_1 - \hat{x}_2 + \hat{x}_3 + l_2 \varepsilon(t), \\ \dot{\hat{x}}_3 & = -q\hat{x}_2 + l_3 \varepsilon(t), \end{cases} \quad (31)$$

Adjustable parameter

$\hat{y}(t) = \hat{x}_1(t), \quad \varepsilon(t) = \bar{y}(t) - \hat{y}(t),$ $L = [l_1, l_2, l_3]^T$ **- gain**

Adaptation algorithm

$$\dot{\hat{\theta}} = \gamma\big(\bar{y}(t) - \hat{y}(t)\big) f_1\big(\bar{y}(t)\big) + \alpha\big(\hat{\theta}_0 - \hat{\theta}(t)\big), \quad (32)$$

γ, α **– parameters,** $\hat{\theta}_0$ **–** ***a priori*** **estimate for** θ**.**

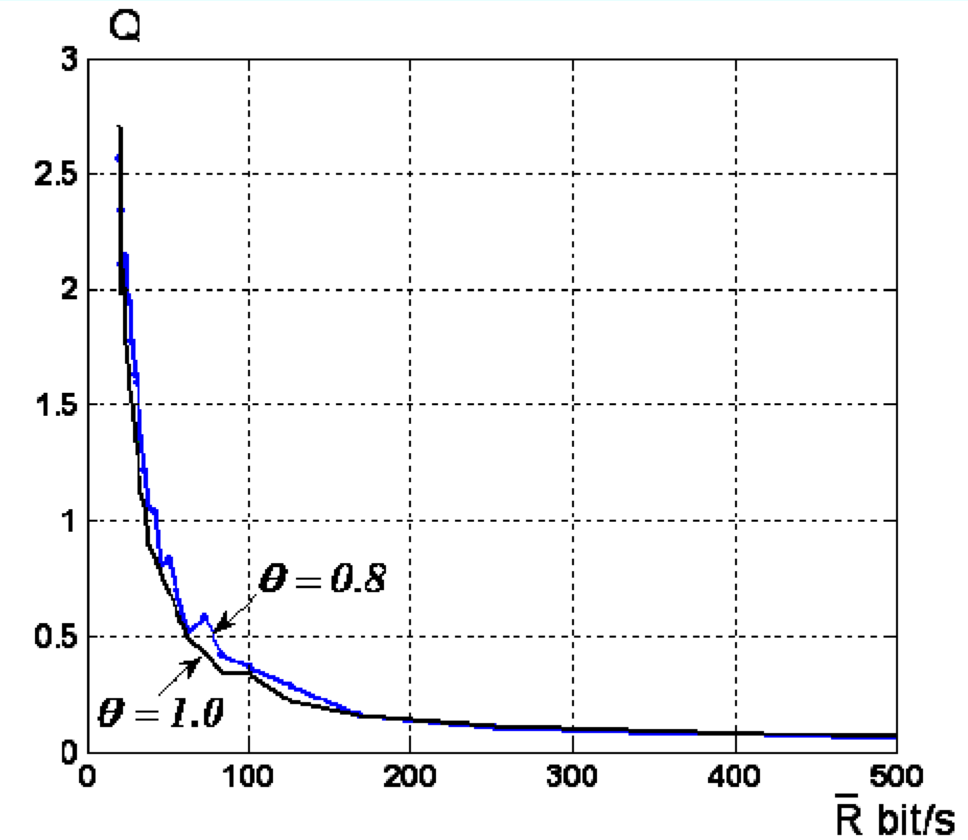
4.4 Simulation results

Parameters: p=10.0, q=15.6, m_0=0.33, m_1=0.22.

Observer gain matrix L = [1.00, 5.54, 4.44]T ; α = 0.02, γ = 0.2.

Two values of θ: θ = 0.8 ; θ = 1.0 .

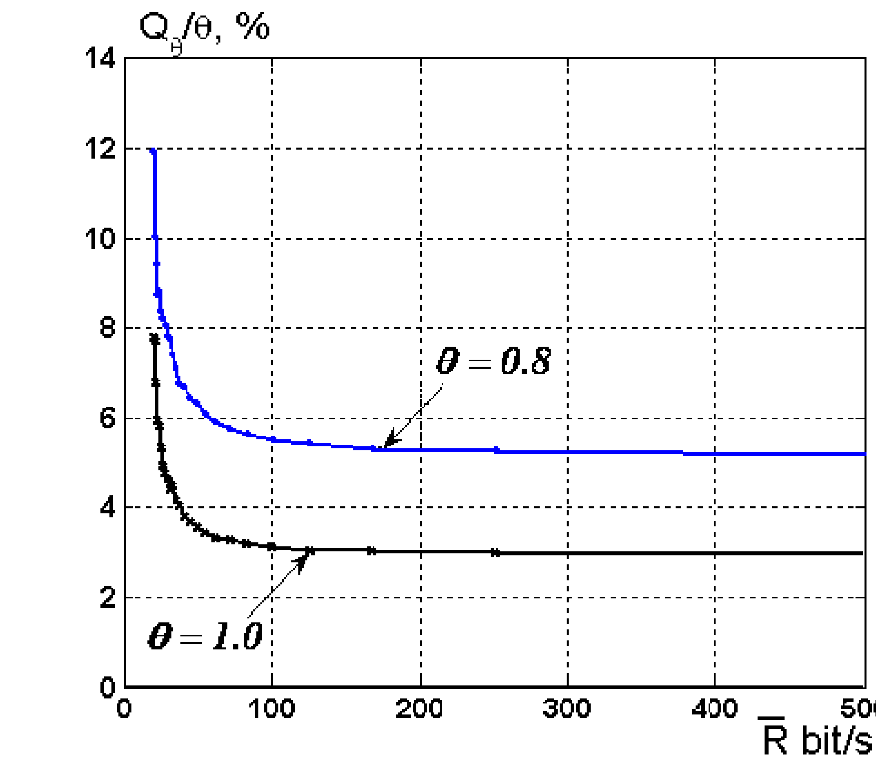
Q vs R for $\hat{\theta}_0 = 0.9$ and different θ.



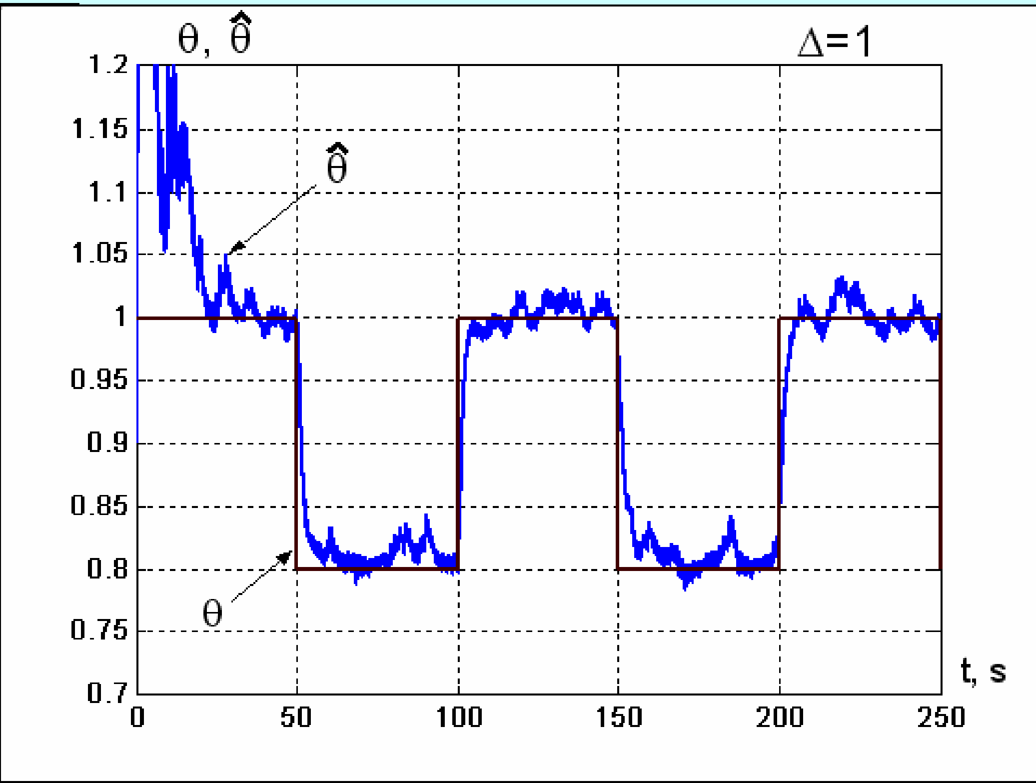
Parameter estimation error

$$Q_\theta = \left(\frac{1}{T}\int_0^T \|\theta - \hat\theta(t)\|^2\,dt\right)^{1/2}$$

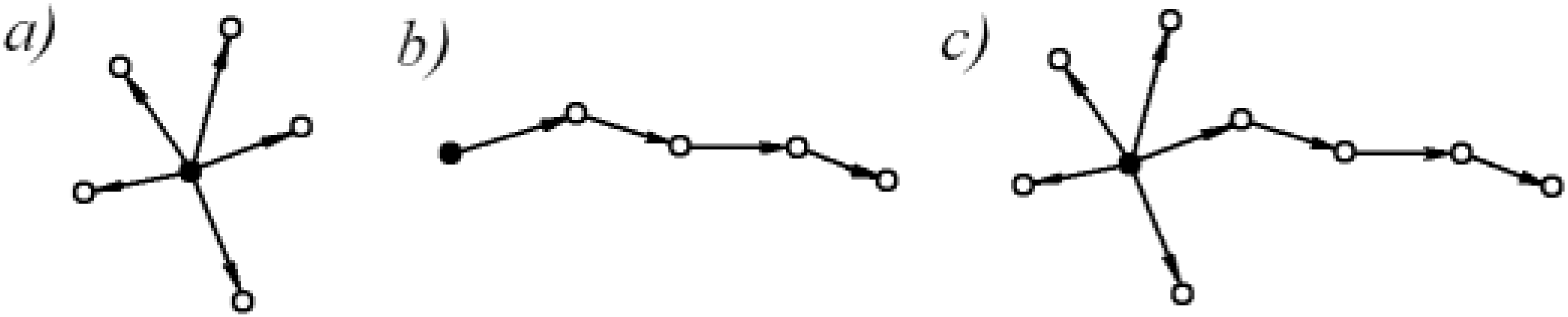
Q_θ/θ* vs *R



Parameter estimation
Δ = 1, *R* = 50 bit/s.



Synchronization of Dynamical Networks under Communication Constraints



"Star", "chain" and "star-chain" network topologies.

A.L. Fradkov, B. Andrievsky. Application of passification method to controlled synchronization of tree networks under information constraints. IEEE MSC'09. St.Petersburg, July, 2009

Limit Synchronization Error

$$\begin{cases} \dot{x}_0 = Ax_0 + \varphi(y_0), \quad y_0 = Cx_0, \\ \dot{x}_1 = Ax_1 + \varphi(\bar{y}_0) + L_1(\bar{y}_0 - y_1), \quad y_1 = Cx_1, \\ \dots \\ \dot{x}_N = Ax_N + \varphi(\bar{y}_{N-1}) + L_N(\bar{y}_{N-1} - y_N), \quad y_N = Cx_N \end{cases},$$

$$Q_i = \overline{\lim_{t\to\infty}} \| x_i(t) - x_0(t) \|,$$

$$Q_i \le K_i / R_i, \text{ where } R_i = \min_{1\le j\le i}\{R_j, R_{cj}\} \quad K_i > 0$$

$\boldsymbol{R_{cj} = v\omega}$ – computation rate

EXTENSIONS

1. Tunable coder may be used to avoid saturation and to improve accuracy.

$$\lambda_k = (\bar{\partial} y_k + \bar{\partial} y_{k-1})/2,$$

$$M_k = m + \begin{cases} \rho M_{k-1}, & \text{if } |\lambda_k| \le 0.5 \\ M_{k-1}/\varrho, & \text{otherwise}, \end{cases}$$

$0 < \varrho \le 1$ – decay parameter; $m = (1-\rho) M_{\min}$,
$M_{\min}$ – minimal value for M_k.

EXTENSIONS-II

2. System structure can be extended beyond Lurie form, e.g.

$$\dot{x} = A(y)x + B\varphi(y), \qquad y = Cx,$$

Then response system is as follows:

$$\dot{\hat{x}} = A(y)\hat{x} + B\varphi(y) + K(\bar{y}(t) - \hat{y}(t)), \qquad \hat{y} = C\hat{x}$$

All results hold under convergence condition, e.g. Demidovich condition for matrix $A_K(y)=A(y)-KC$**, namely**

$$\lambda_i(A_K(y) + A_K(y)^T) \leq -\varepsilon < 0, \qquad i = 1, 2, \ldots, n$$

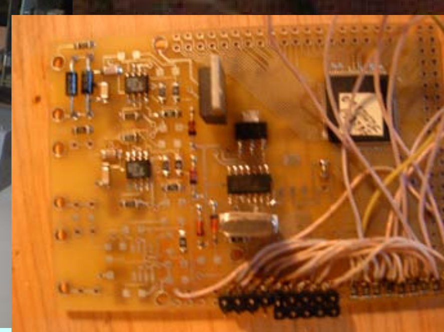
CONCLUSIONS

• For systems with first-order coders the upper and lower bounds of limit synchronization error are proportional to the maximum rate of the coupling signal and inversely proportional to the data transmission rate (channel capacity).

• If channel bit-rate is sufficiently large then an arbitrarily small limit synchronisation error can be reached for uncertain system by means of adaptive synchronization algorithm.

•More results for nonadaptive case: see survey
•Andrievsky, Matveev, Fradkov. Automation and Remote Control, 2010, 4.

Experimental Multi-Pendulum Set-Up in IPME, St.Petersburg



CONCLUSIONS

1) Decentralized adaptive output feedback synchronization **problem for nonlinear networks encompassing coordinate synchronization and consensus problem** is studied.

2) In contrast to a large number of previous results, we obtained control algorithms and synchronization conditions for networks consisting of **nonidentical nonlinear systems with incomplete measurement, incomplete control, incomplete information about system parameters and coupling.**

3) Control algorithm design is based on **speed-gradient method, passivity, passification.**

Future research: examination of the influence of information constraints with full order coder.

Thank

you

Tool: Decentralized Speed-Gradient approach

(M.V. Druzhinina and A. L. Fradkov. Adaptive decentralized control of interconnected systems. Proc. of 14th IFAC WC, 1999, Vol.L, pp.175-180,

Fradkov A.L., Miroshnik I.V., Nikiforov V.O. Nonlinear and Adaptive Control of Complex Systems. Dordrecht: Kluwer Academic Publ., 1999).

$$\dot{x}_i = f_i(x_i) + b_i(x_i, u_i) + h_i(x), \quad i = 1 \ldots N, \tag{1}$$

$$h_i(x) \le \sum_{j=1}^{N} \xi_{ij} ||x_j||, \quad \xi_{ij} \ge 0, \quad i = 1 \ldots N. \tag{2}$$

Control Goal:

$$Q_i(x_i(t)) \to 0 \; as \; t \to \infty, \tag{5}$$

Assumption A1: local subsystems

$$\dot{x}_i = f_i(x_i) + b_i(x_i, u_i), \quad i = 1 \dots N, \qquad (6)$$

can be exponentially stabilized, i.e. there exist *$u_i^*(x_i)$*:

$$\alpha_{1i}||x_i||^2 \le Q_i(x_i) \le \alpha_{2i}||x_i||^2$$

$$||\nabla Q_i(x_i)|| \le \beta_i ||x_i|| \qquad (7)$$

$$\dot{Q}_i(x_i, u_i)\Big|_{u_i = u_i^*(x_i)} \le -\rho_i Q_i(x_i),$$

where

$$\dot{Q}_i(x_i)\Big|_{u_i = u_i^*(x_i)} =$$

$$= \nabla Q_i(x_i)^T \left(f_i(x_i) + b_i(x_i, u_i^*(x_i))\right),$$

Theorem. Let: 1. Local laws $u_i^*(x_i)$ can be represented as smooth functions of auxiliary parameters:

$$u_i^*(x_i) = U_i(x_i, \theta_i^*), \quad i = 1 \ldots N,$$

2. Time-derivative of the function $Q_i(x_i)$ is convex in θ_i, i.e. any θ_i, θ_i', x_i satisfy the inequality:

$$\dot{Q}_i(x_i, \theta_i') - \dot{Q}_i(x_i, \theta_i) \geq (\theta_i - \theta_i')^T \nabla_{\theta_i} \left(\dot{Q}_i(x_i, \theta_i) \right)$$

3. Functions $h_i(\cdot), \quad i = 1 \ldots N$ satisfy (2) and additional condition:

$$\beta_i^2 + \sum_{j=1}^{N} \xi_{ij}^2 \; < \; 2\rho_i \alpha_{1i} / \sqrt{N}, \quad i = 1 \ldots N. \quad (13)$$

Then the decentralized adaptive control law

$$
\begin{aligned}
u_i &= U_i(x_i, \theta_i) \\
\dot{\theta}_i &= -\gamma_i \nabla_{\theta_i} \left(\nabla Q_i^T b_i(x_i, U_i(x_i, \theta_i)) \right), \qquad (14) \\
& \quad i = 1 \ldots N,
\end{aligned}
$$

where $\gamma_i > 0, \quad i = 1 \ldots N$, provides the boundedness of the whole system (1),(14) trajectories and the achievement of the goal (5).

Proof. Choose Lyapunov function and evaluate its time derivative:

$$V_a(x,\theta) = \sum_{i=1}^{N} V_{ia}(x_i,\theta_i) = \sum_{i=1}^{N}\{Q_i(x_i) + \frac{1}{2\gamma_i}(\theta_i - \theta_i^*)^T(\theta_i - \theta_i^*)\}$$

$$\dot V_a = \sum_{i=1}^{N}\{\dot Q_i(x_i,\theta_i) - (\theta_i - \theta_i^*)^T \nabla_{\theta_i}(\dot Q_i) +$$

$$+\nabla Q_i^T h_i(x)\} \le \sum_{i=1}^{N}\{\dot Q_i(x_i,\theta_i^*) + \nabla Q_i^T h_i(x)\}.$$

Then apply A1 and condition (2):

$$\dot V_a \le -\sum_{i=1}^{N}\rho_i Q_i(x_i) + \sum_{i=1}^{N}\beta_i||x_i||\sum_{j=1}^{N}\xi_{ij}||x_j|| \le$$

$$\le -\sum_{i=1}^{N}\alpha_{1i}\rho_i||x_i||^2 + \sum_{i=1}^{N}\beta_i||x_i||\sum_{j=1}^{N}\xi_{ij}||x_j||.$$

Introduce notation:

$$\eta_i \triangleq \frac{\sqrt{N}}{2}\left(\beta_i^2 + \sum_j \xi_{ij}^2\right) - \rho_i. \tag{15}$$

Choosing $h_i(\cdot)$, which satisfy the inequality (13), we obtain that $\eta_i > 0$. Thus, we have:

$$\dot{V}_a \le -\sum_{i=1}^{N} \eta_i ||x_i||^2 \le -\sum_{i=1}^{N} \frac{\eta_i}{\alpha_{2i}} Q_i(x_i), \tag{16}$$

Integrate (16) over [0,t]:

$$0 \le \sum_{i=1}^{N} Q(x_i(t)) \le V_a(x(t), \theta(t)) \le$$

$$\le V_a(x(0), \theta(0)) - \sum_{i=1}^{N} \frac{\eta_i}{\alpha_{2i}} \int_0^t Q_i(x_i(s))ds,$$

The boundedness of the system (1), (14) trajectories $x(t), \theta(t)$ follows from the growth condition (11) for objective functions $Q_i(\cdot), \quad i = 1, \ldots, N$.

Taking into account boundedness of the right hand sides of the system (1), (14),one may show that $x_i(t)$ are uniformly continuous. Hence functions $Q_i(x_i(t)), \quad i = 1 \ldots N$, are uniformly continuous and integrable on $[0, \infty)$, i.e. satisfy the condition of Barbalat's lemma, which guarantees that $Q_i(x_i(t)) \to 0$ when $t \to \infty$. This proved the theorem.