Suboptimality estimates for NMPC schemes

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in collaboration with

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"Control theory for digitally networked dynamical systems"

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Setup

We consider nonlinear discrete time control systems

$$x(n+1) = f(x(n), u(n))$$

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Problem: Optimal stabilization via infinite horizon optimal control:

For a running cost $\ell : X \times U \to \mathbb{R}_0^+$ solve

$$\text{minimize} \;\; J_\infty(x,u) = \sum_{n=0}^{\infty} \ell(x(n), u(n))$$

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We obtain a feedback law F_N by a moving horizon technique

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At each time instant τ solve for the current state x_τ

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$\rightsquigarrow$ optimal control $u^*(0), \ldots, u^*(N-1)$

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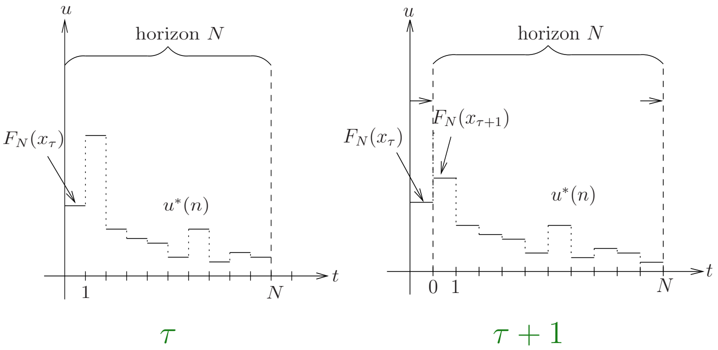
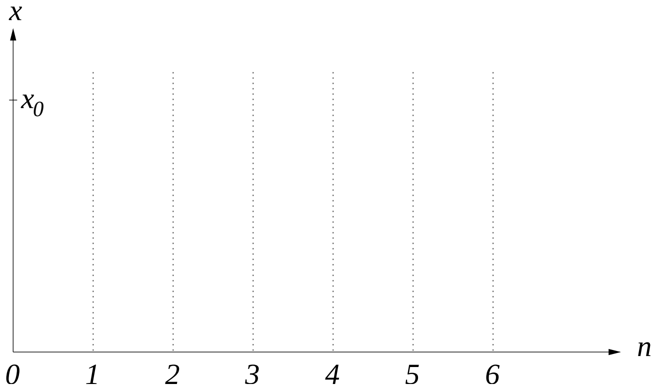


MPC from the trajectory point of view



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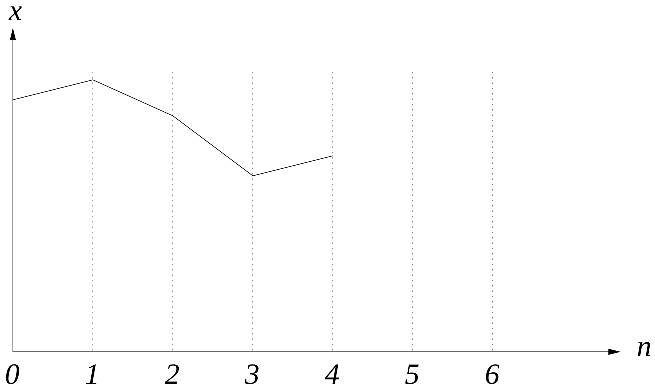

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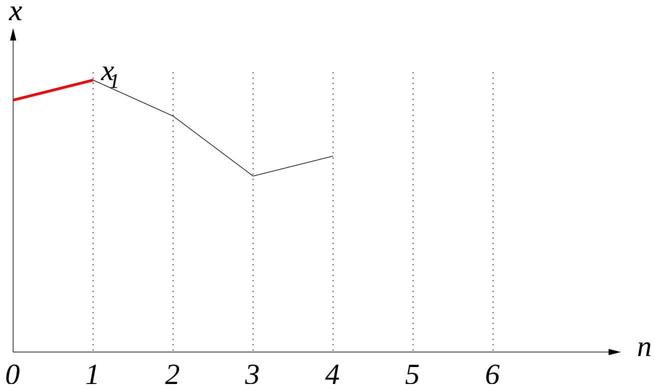

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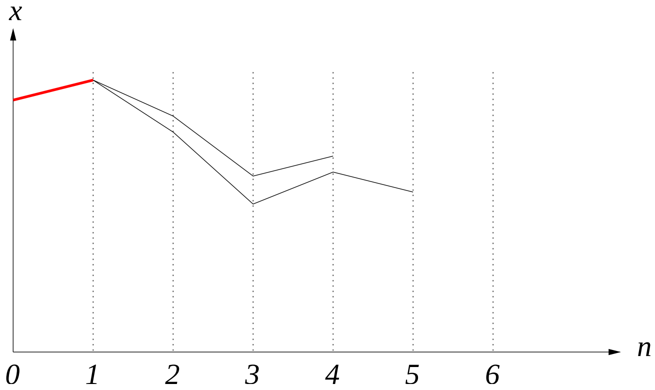

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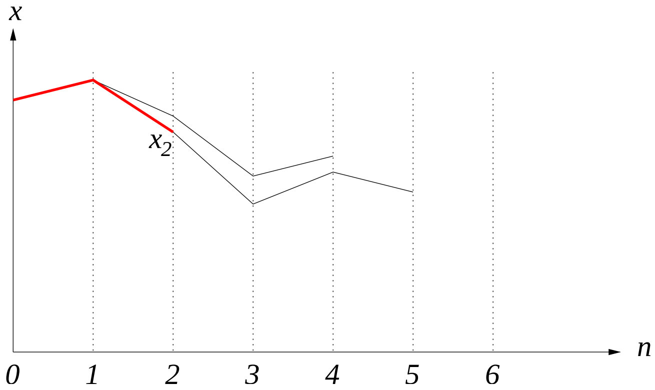

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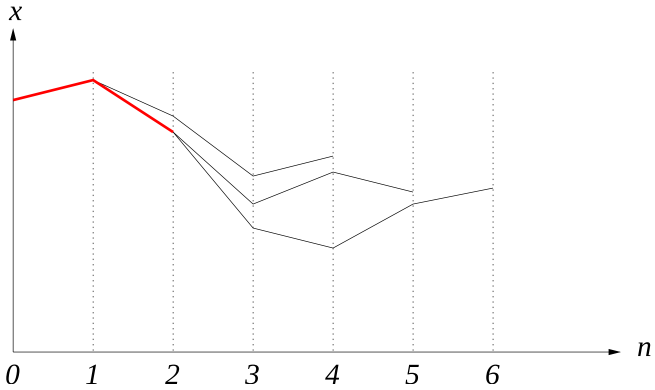

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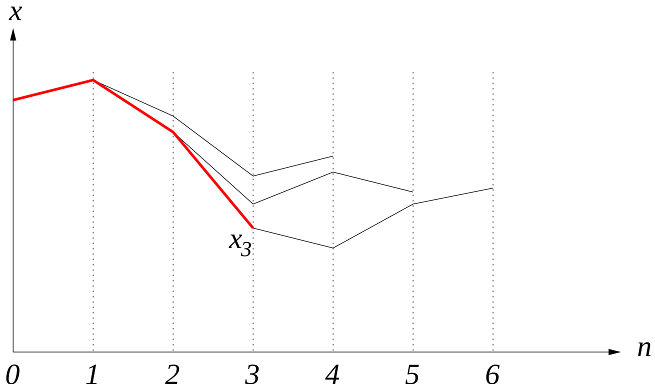

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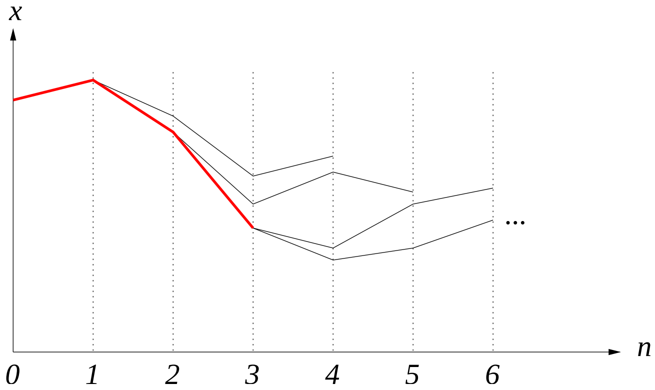

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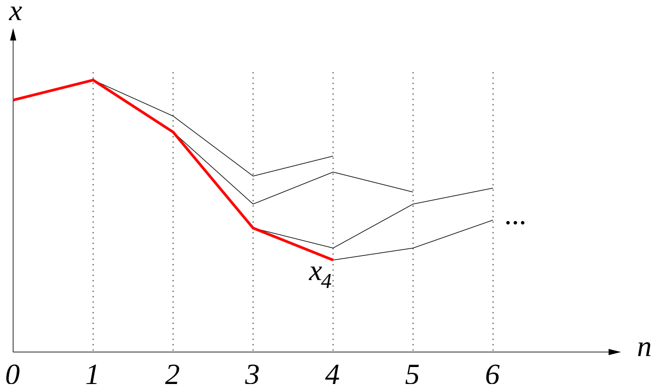

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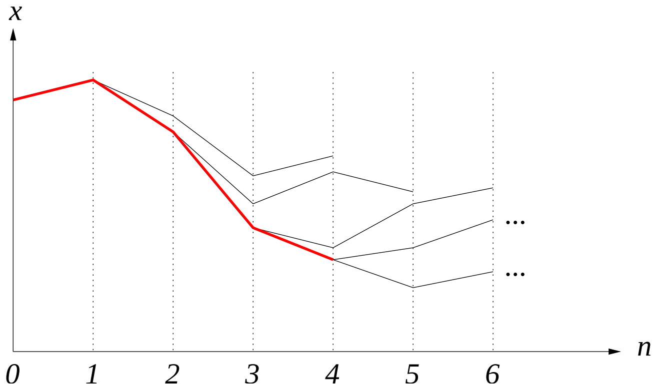

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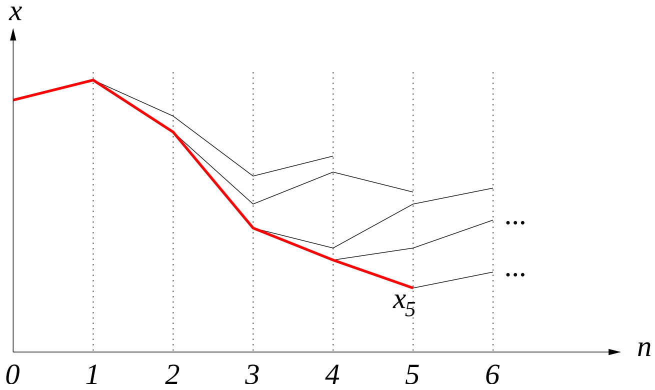

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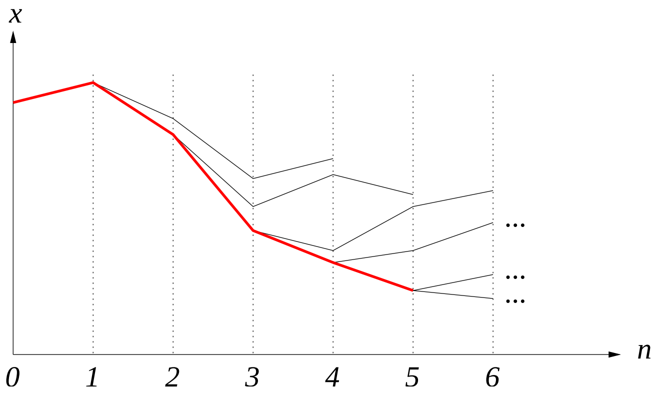

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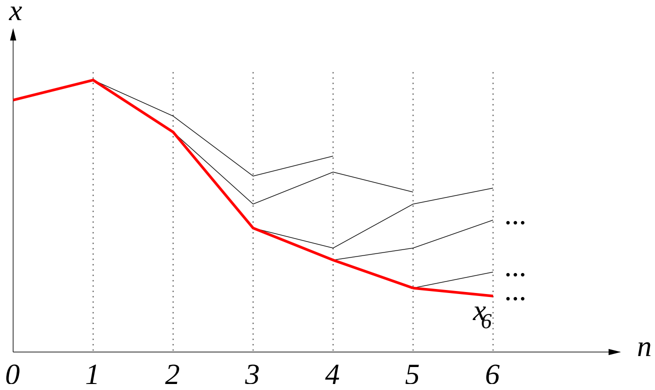

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Relaxed dynamic programming inequality

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Theorem: If there exists $\alpha \in (0,1]$ such that the relaxed dynamic programming inequality

$$V_N(f(x, F_N(x))) \leq V_N(x) - \alpha \ell(x, F_N(x))$$

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Note: The last inequality does not hold for MPC schemes with stabilizing terminal constraints

Computing α

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under the C, σ-exponential controllability property (with respect to the running cost l):

For each $x(0)$ there exists a control sequence $u(\cdot)$ such that

$$\ell(x(n), u(n)) \leq C\sigma^n \ell^*(x(0))$$

holds for all $n \in \mathbb{N}_0$, where $\ell^*(x(0)) = \inf_{u \in U} \ell(x(0), u)$

Suboptimality and stability condition

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Theorem: The value

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Furthermore: If $\alpha(C,\sigma) < 0$ then there exists a C, σ-exponentially controllable system, which is not stabilized by F_N.

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- the bounds on the $\ell(x^*(n), u^*(n))$ and the controllability condition induce upper bounds on $V_N(f(x, F_N(x)))$
- combining these bounds leads to a linear program for α
- this linear program is explicitly solvable
- the converse statement for $\alpha(C, \sigma) < 0$ is obtained by explicit construction of a counterexample

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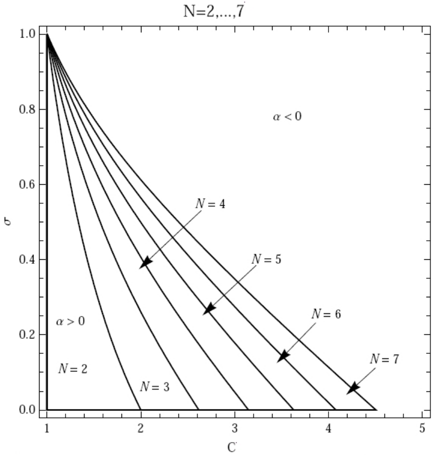
Stability chart for C and σ

Minimal horizon N for stable NMPC depending on C and σ



(Figure: Harald Voit)

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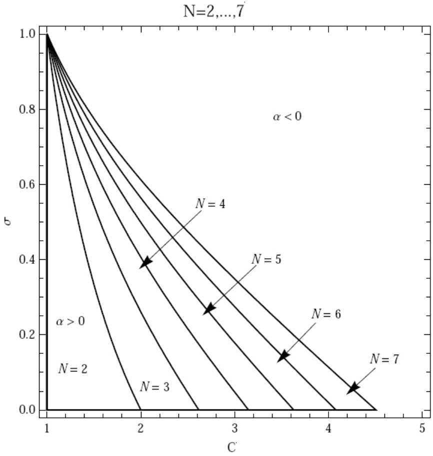

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Conclusion: good performance can be expected for small overshoot C

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We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

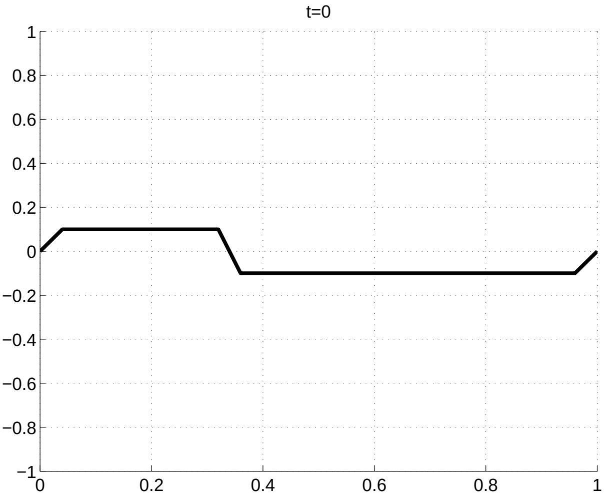
domain $\Omega = [0, 1]$

solution $y = y(t, x)$ and distributed control $u = u(t, x)$

boundary conditions $y(t, 0) = y(t, 1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

The uncontrolled PDE



uncontrolled ($u \equiv 0$)

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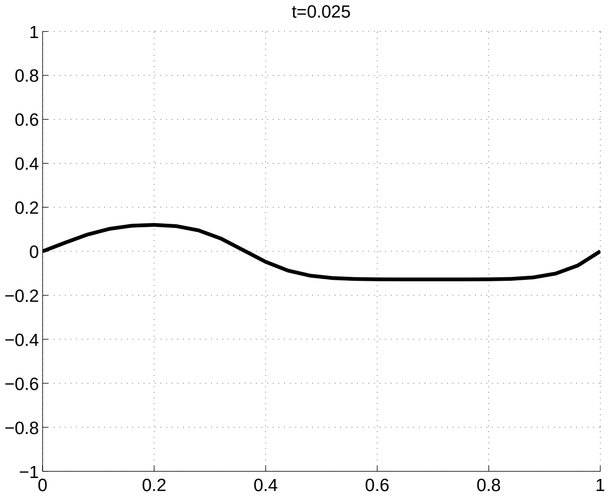

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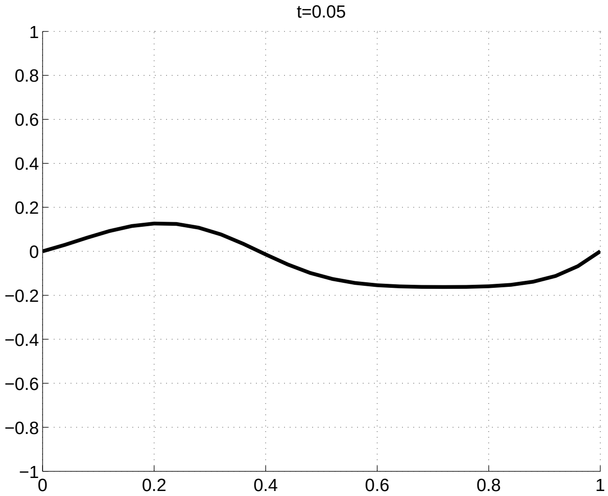

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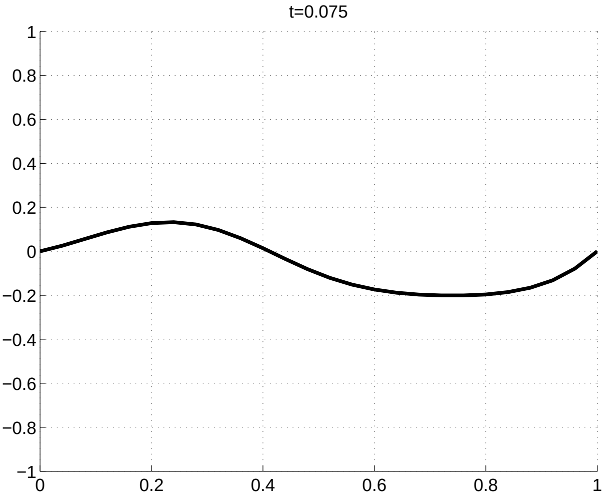

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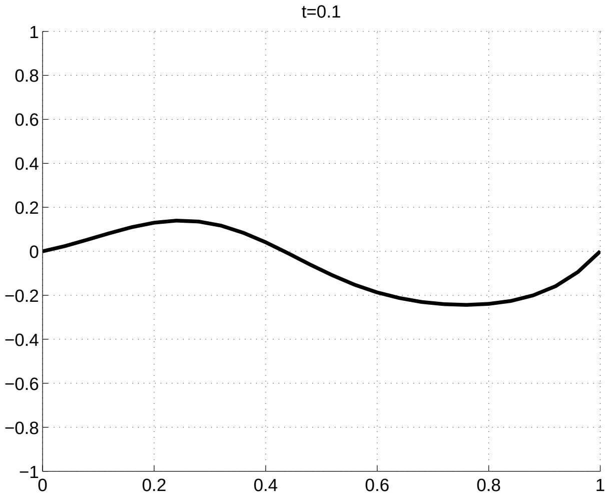

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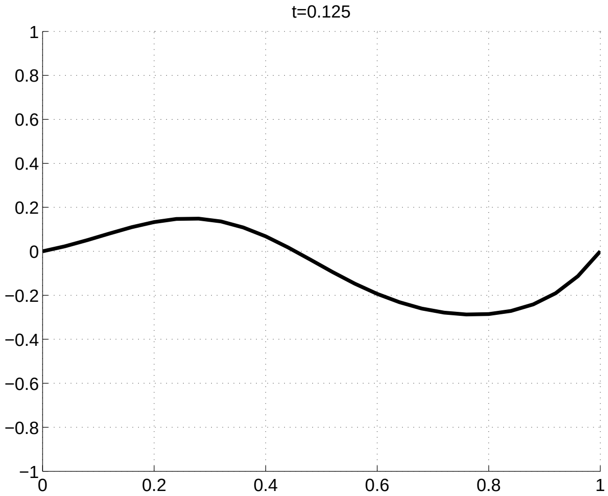

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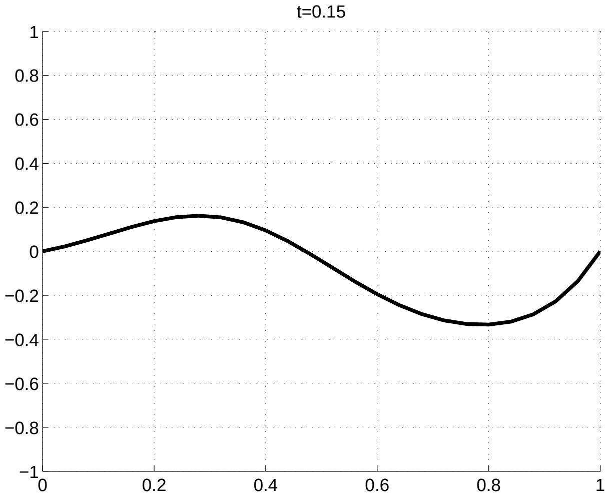

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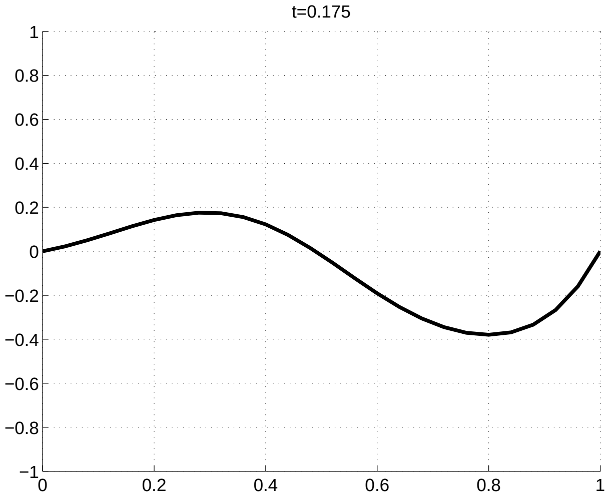

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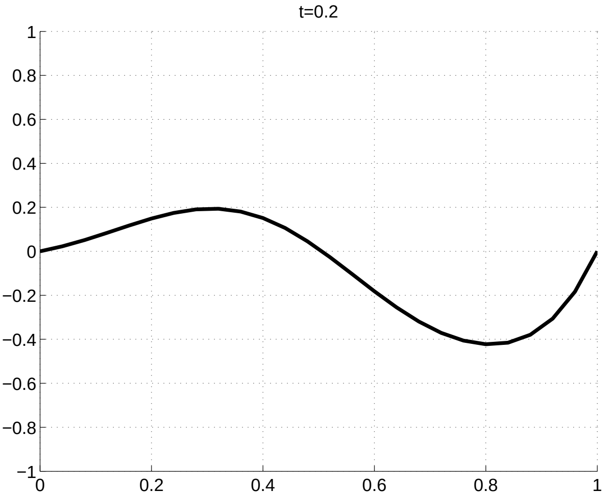

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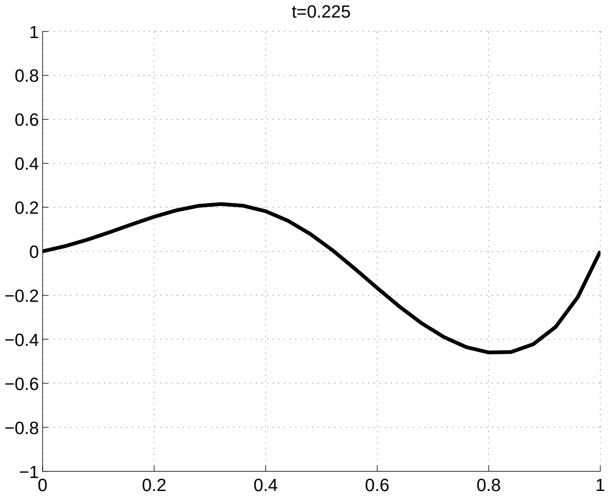

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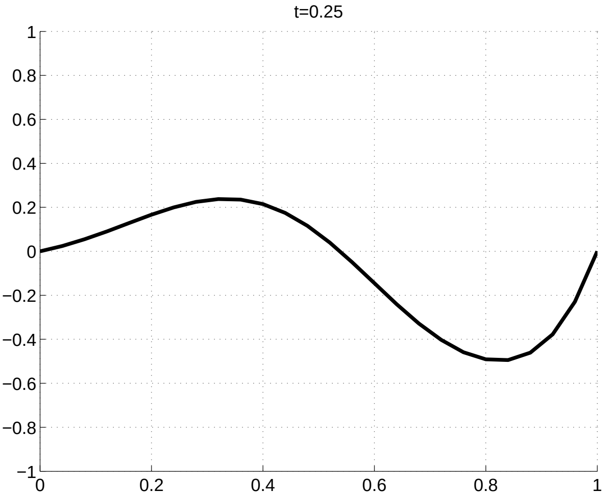

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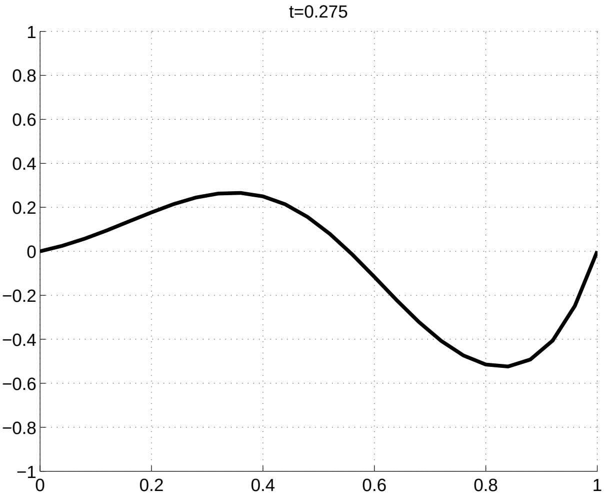

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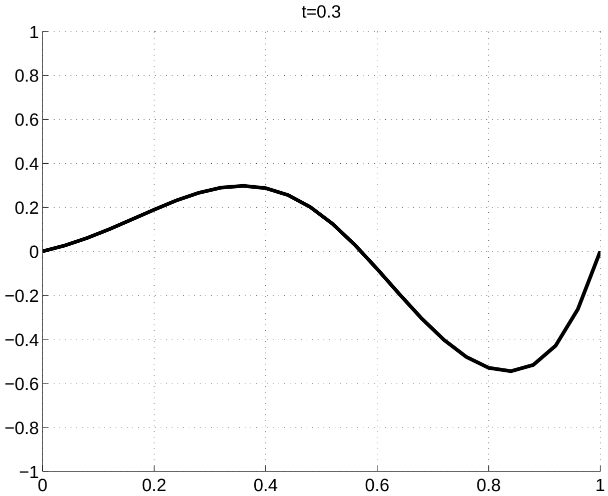

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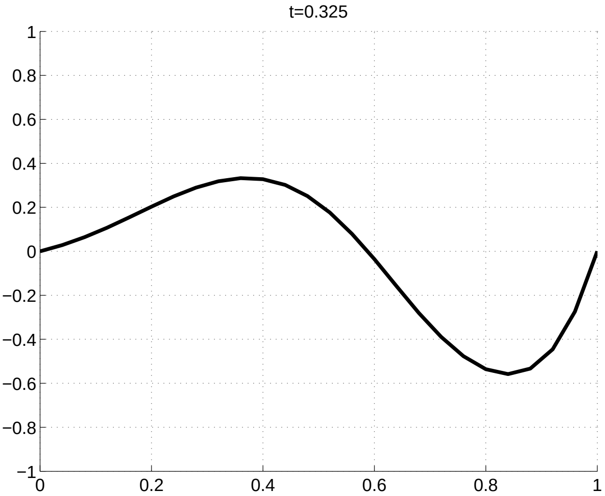

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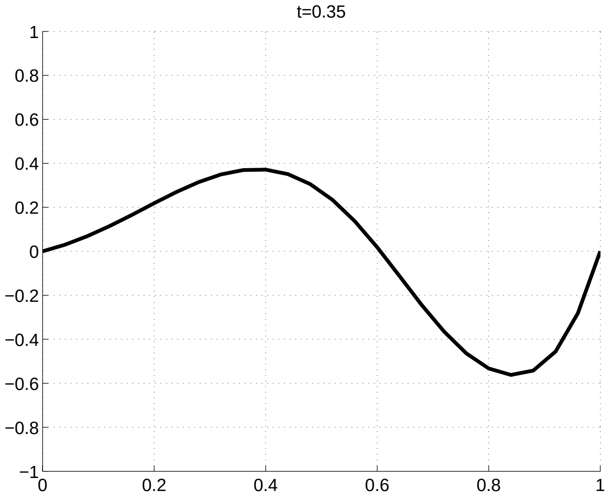

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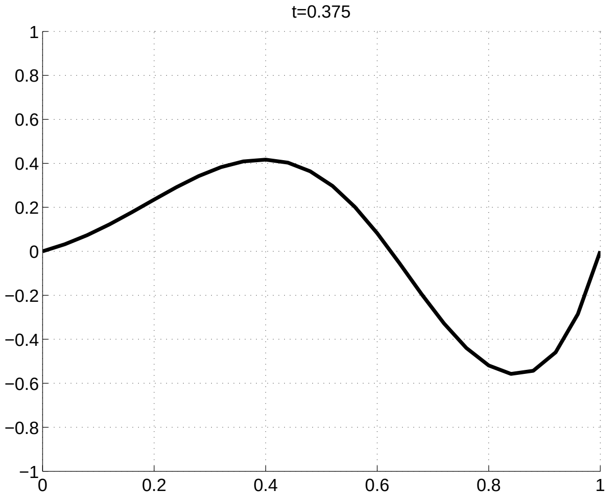

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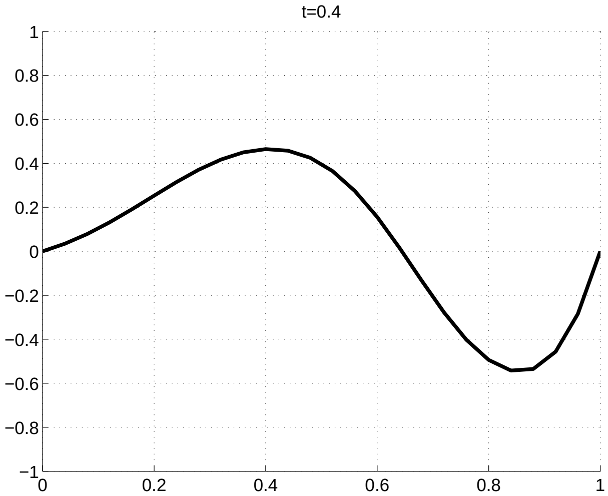

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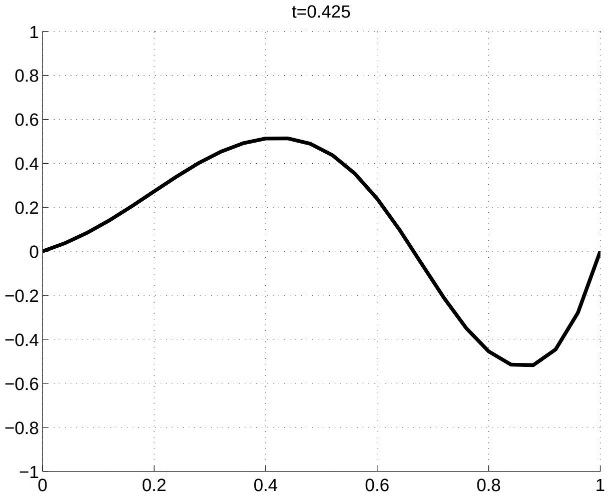

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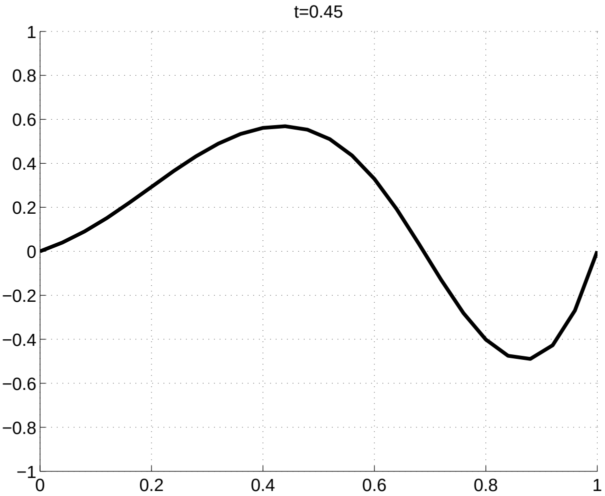

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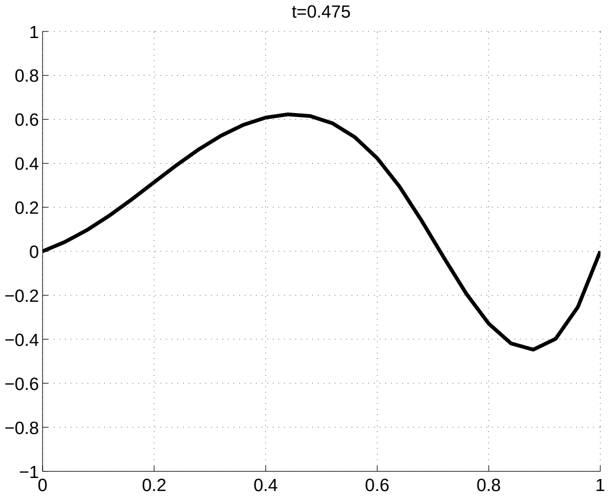

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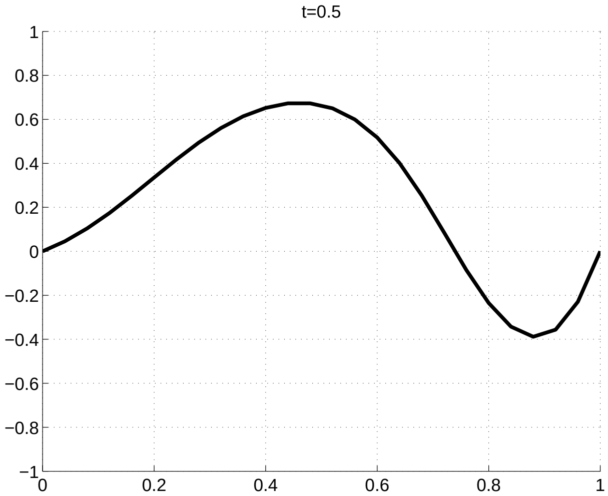

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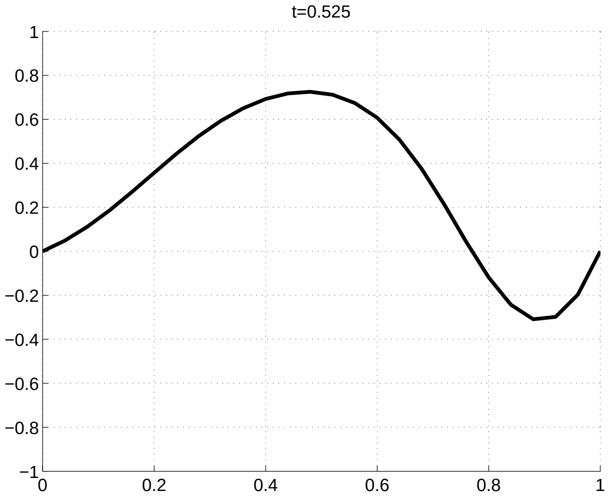

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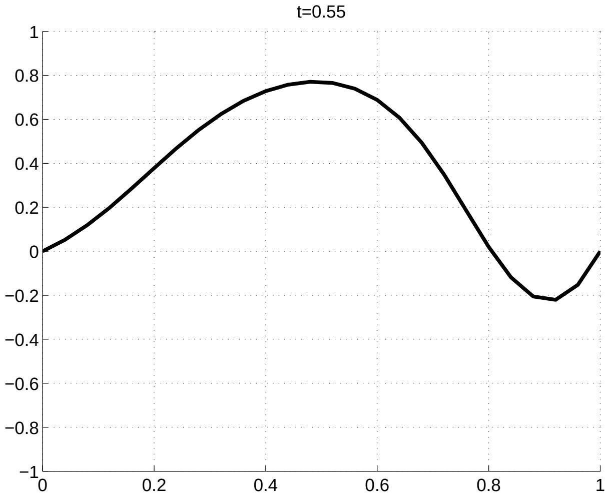

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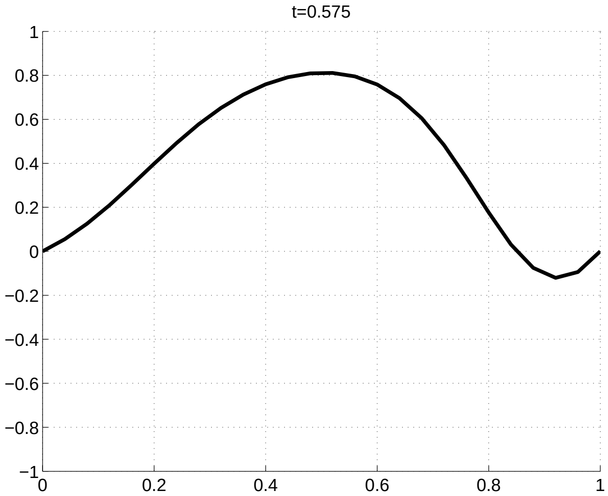

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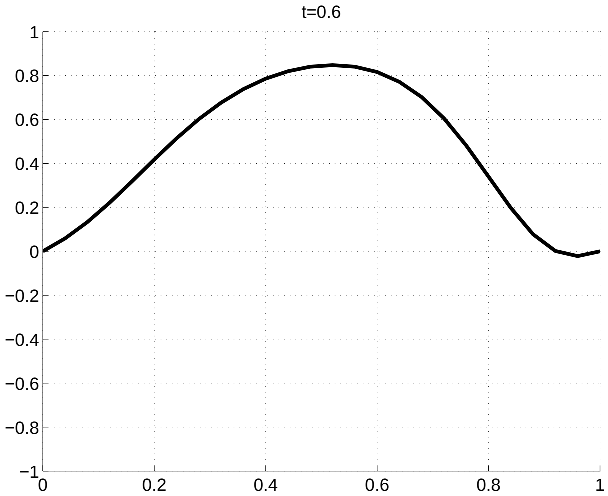

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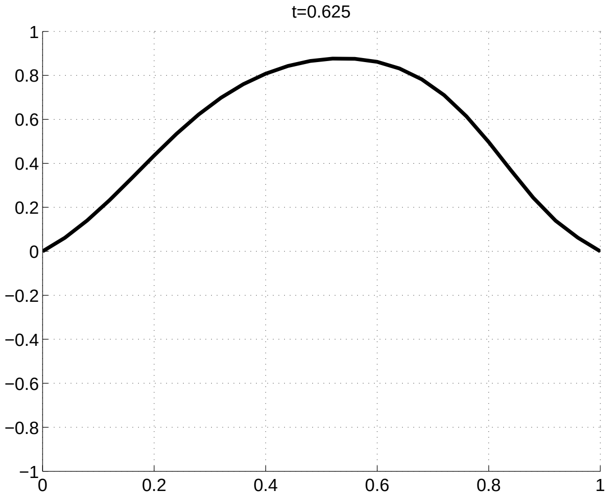

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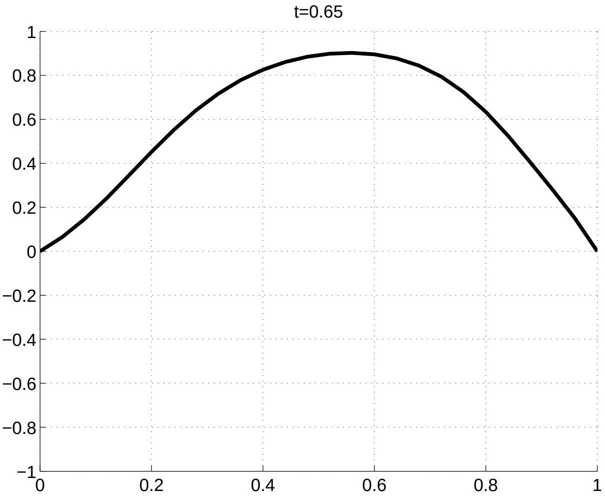

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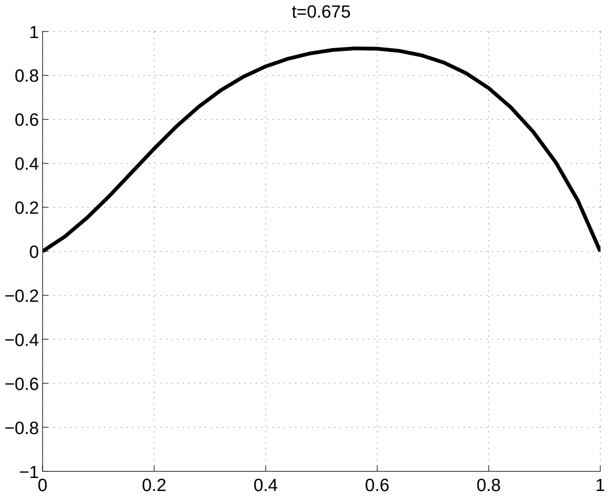

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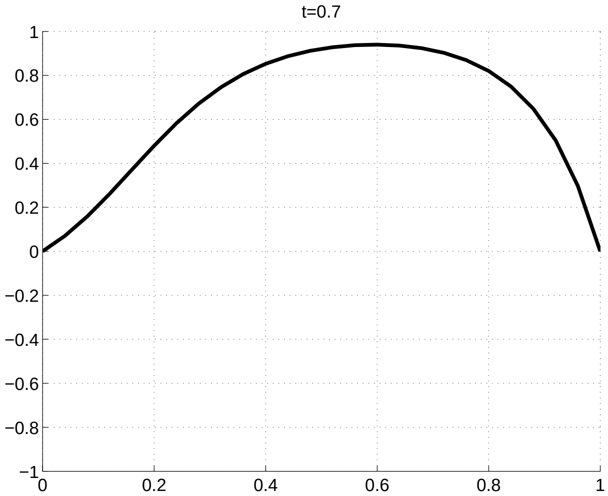

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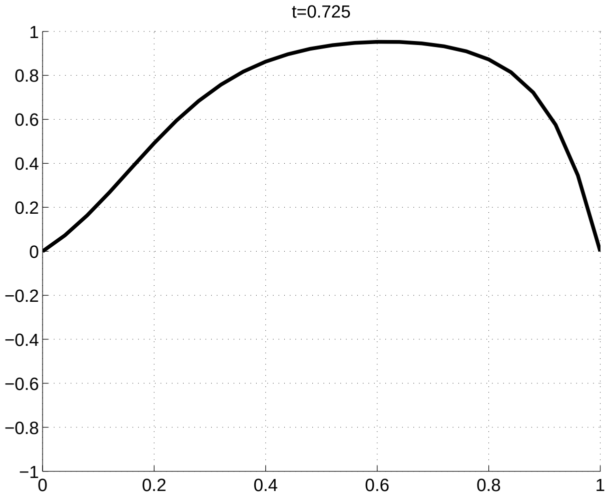

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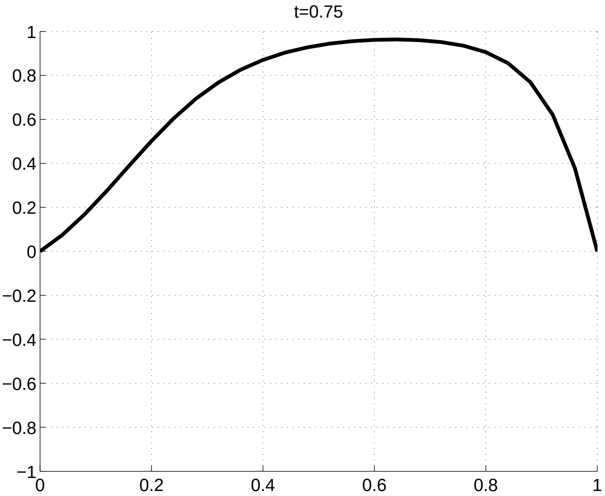

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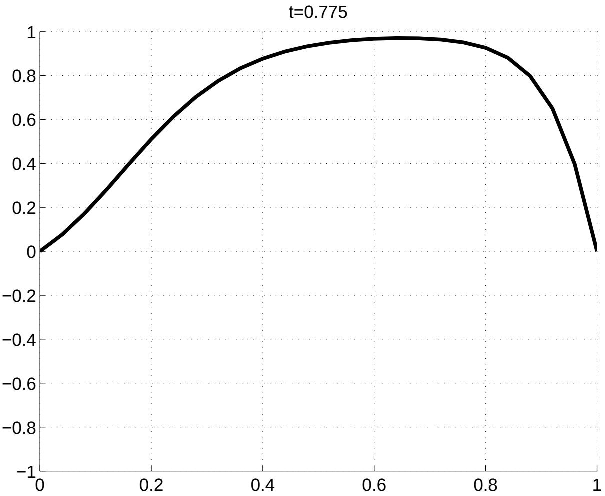

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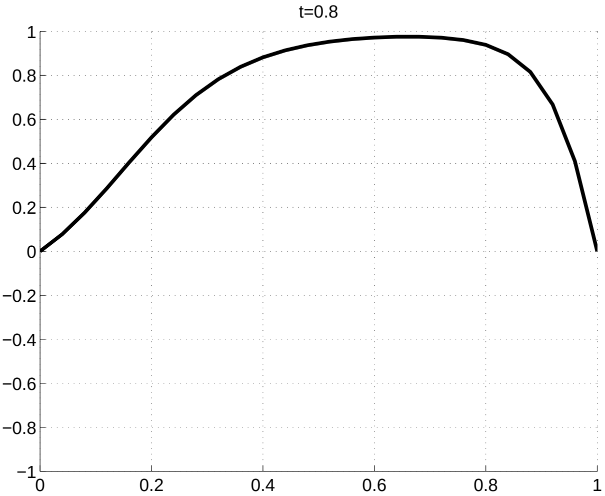

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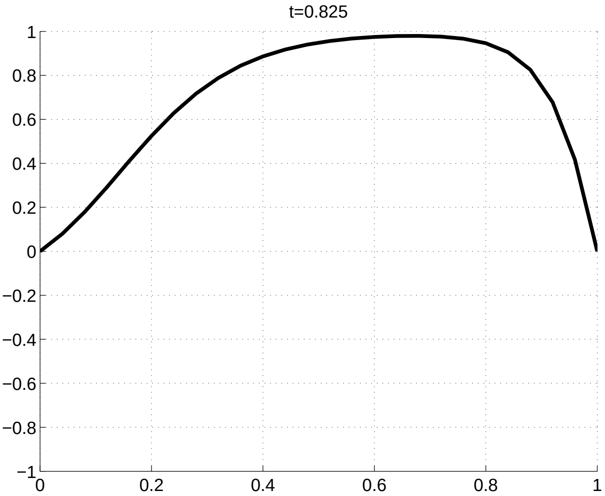

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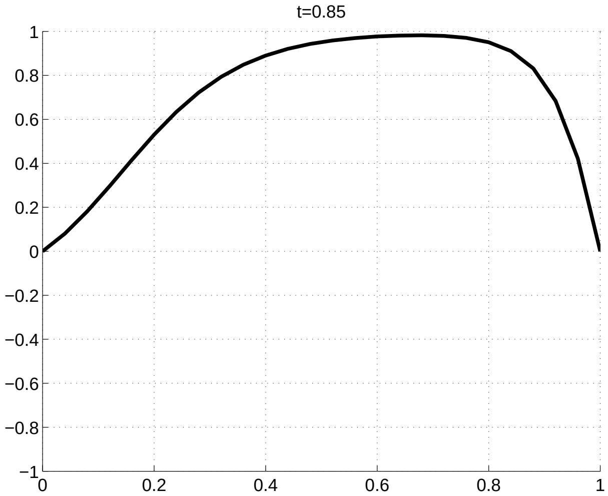

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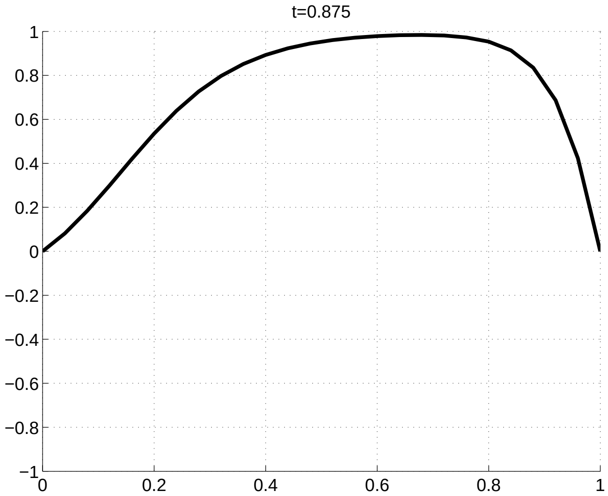

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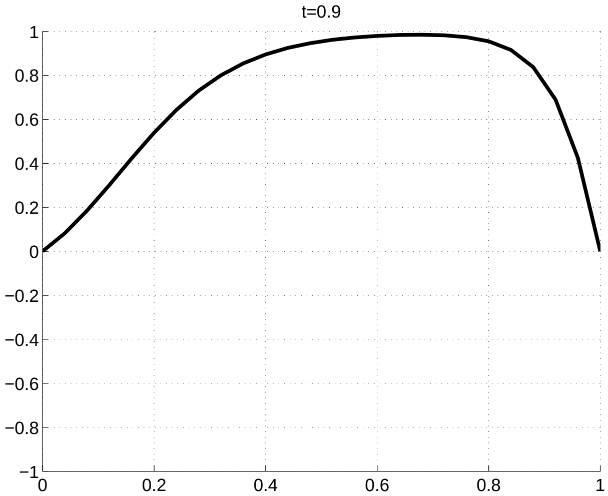

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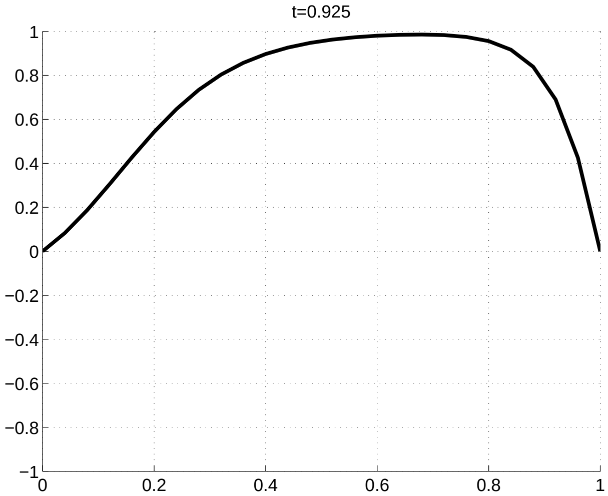

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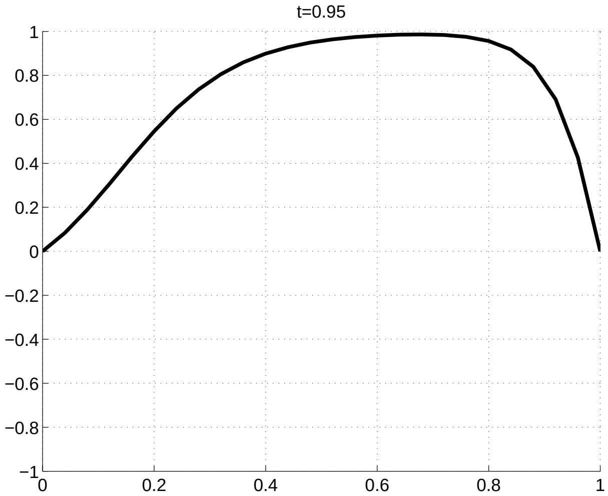

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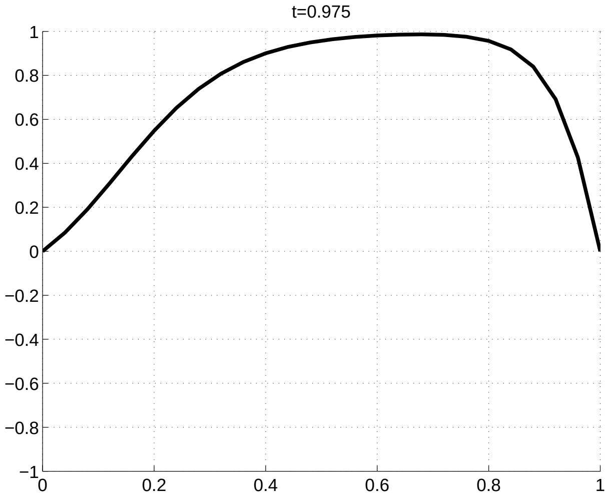

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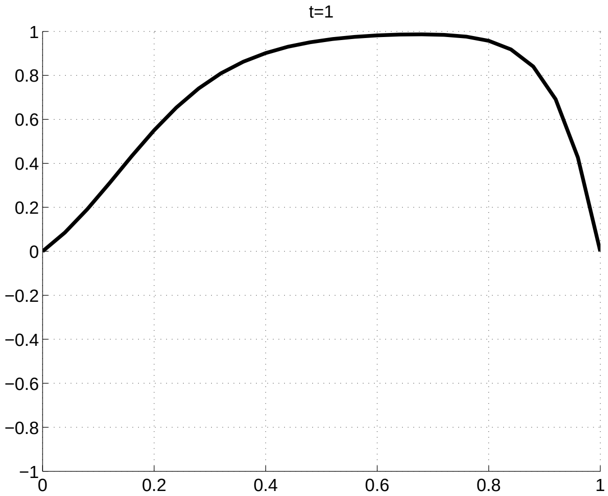

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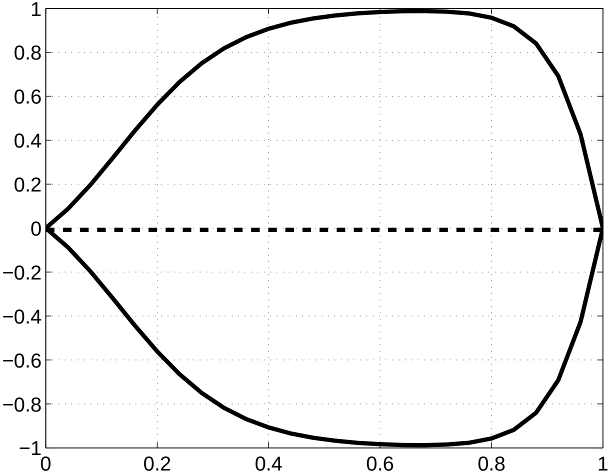

all equilibrium solutions

MPC for the PDE example

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

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Usual cost: quadratic L^2 cost

$$\ell(y(n,\cdot), u(n,\cdot)) = \|y(n,\cdot)\|_{L^2}^2 + \lambda \|u(n,\cdot)\|_{L^2}^2$$

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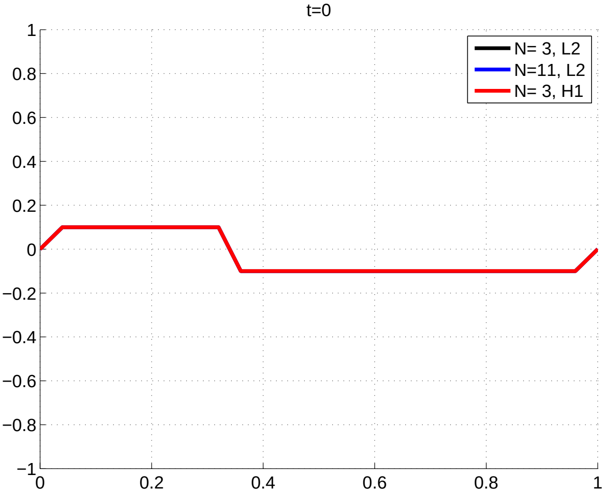
A small calculation reveals that for this L^2 cost the overshoot C is much larger than for the H^1 cost

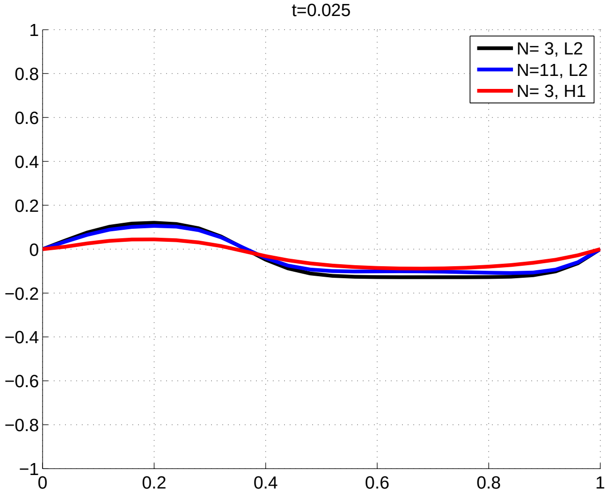
$$\ell(y(n,\cdot),u(n,\cdot)) = \underbrace{\|y(n,\cdot)\|_{L^2}^2 + \|y_x(n,\cdot)\|_{L^2}^2}_{=\|y(n,\cdot)\|_{H^1}^2} + \lambda\|u(n,\cdot)\|_{L^2}^2.$$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

MPC with L_2 vs. H_1 cost



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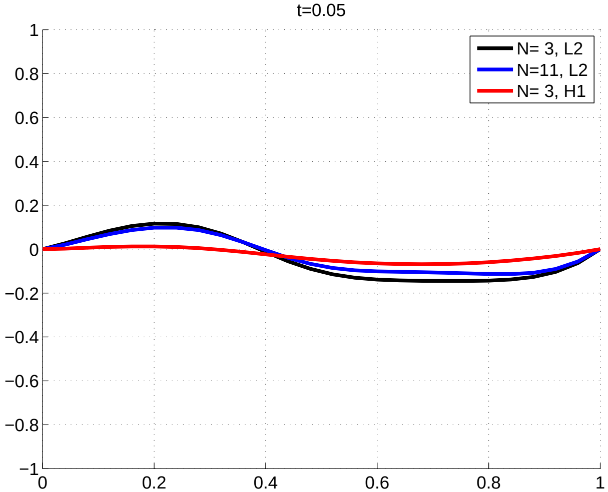

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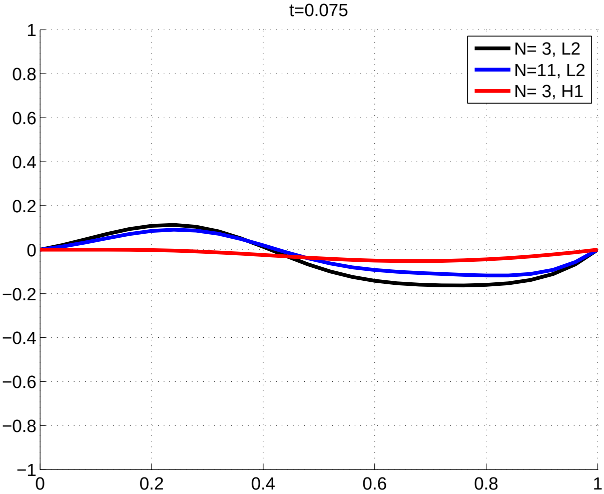

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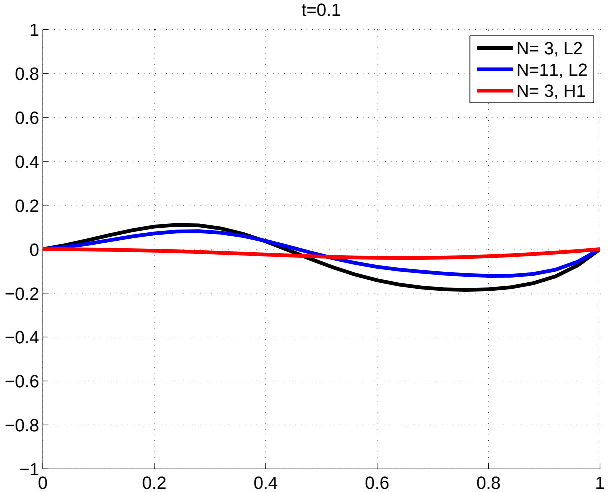

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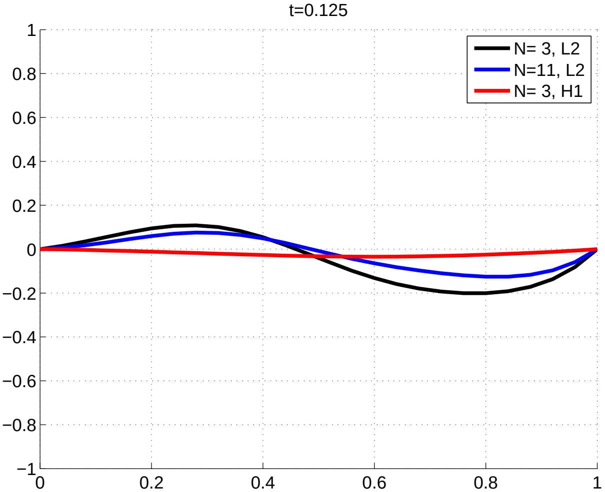

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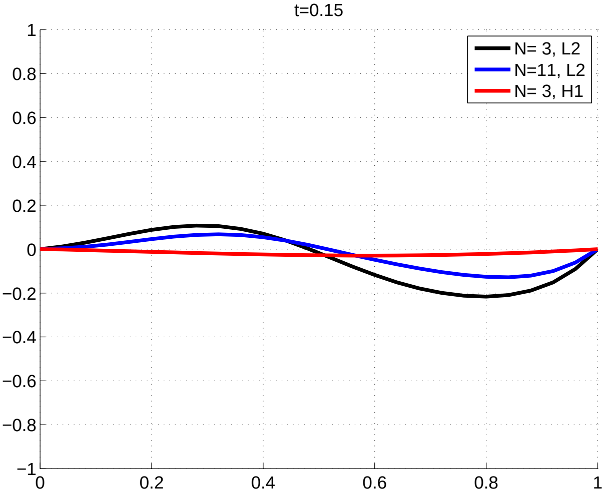

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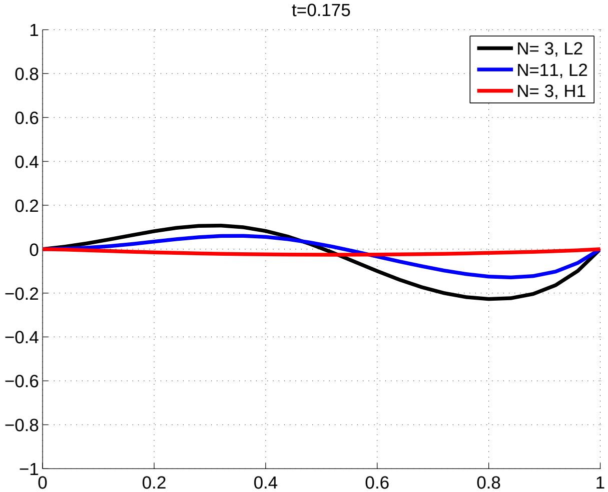

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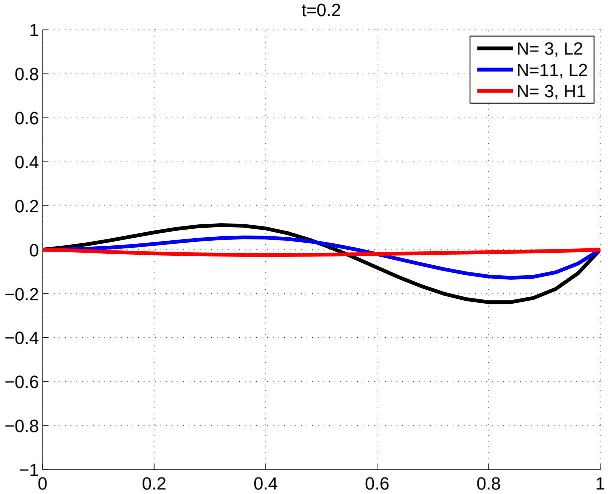

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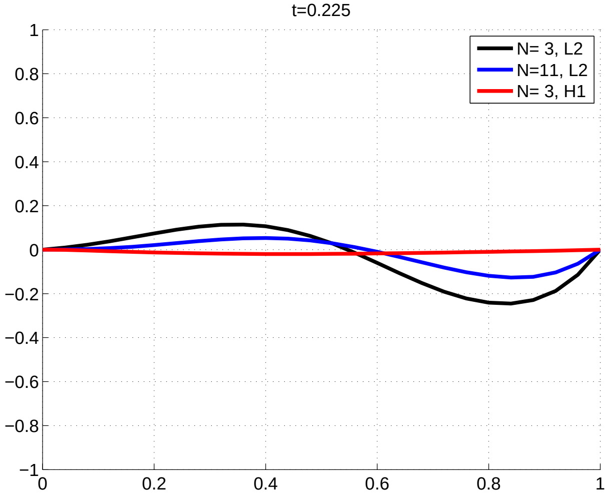

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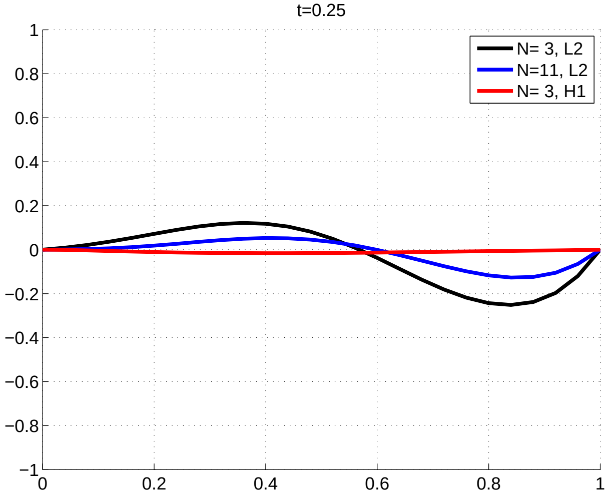

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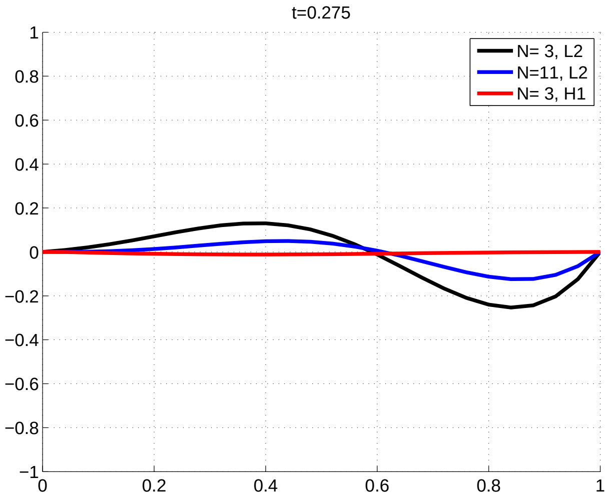

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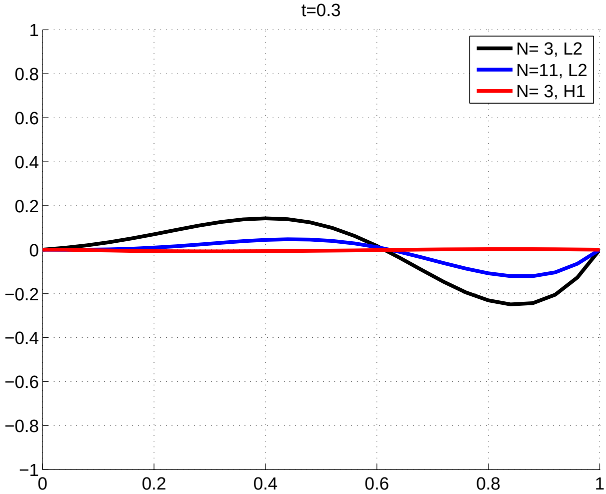

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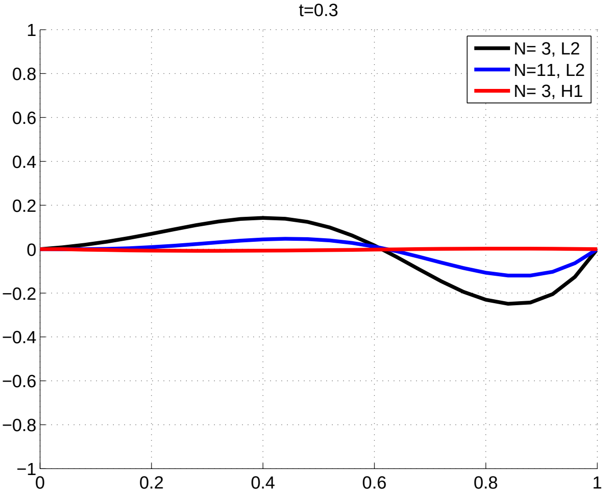

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similar results can be obtained for the wave equation

Networked MPC

We now consider an MPC controller implemented via a network

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For simplicity of exposition, we neglect

- computation and transmission delay
- disturbance and prediction error

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Both can be added to our analysis, cf. [Gr./Pannek/ Worthmann '09] and [Findeisen/Gr., in preparation]

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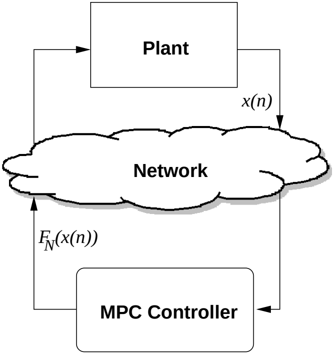
Here, we only consider packet loss

Packet loss



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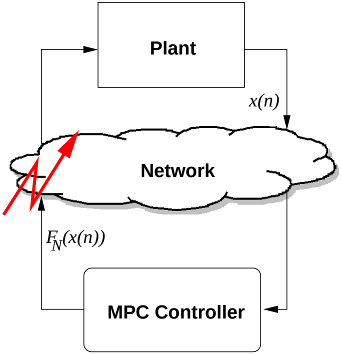

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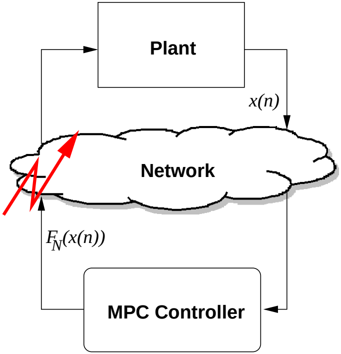
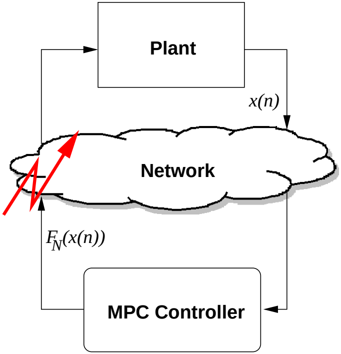


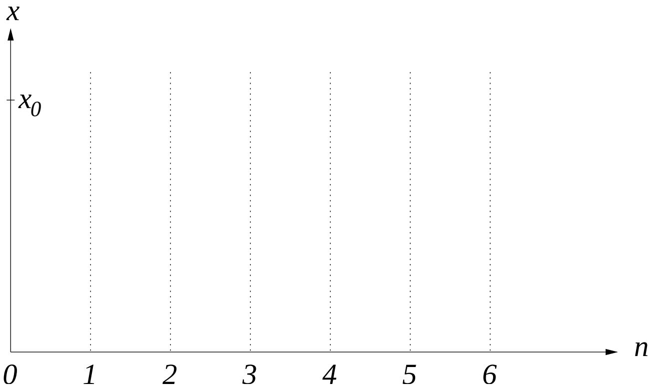
Idea: • send several values of optimal open loop control sequence (instead of just the first value)

Packet loss



Idea:
- send several values of optimal open loop control sequence (instead of just the first value)
- use these values until next values arrive

Schematic illustration of the idea



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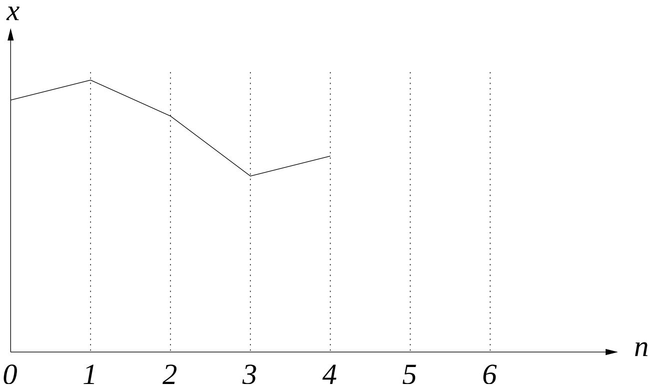

black = predictions (open loop optimization)

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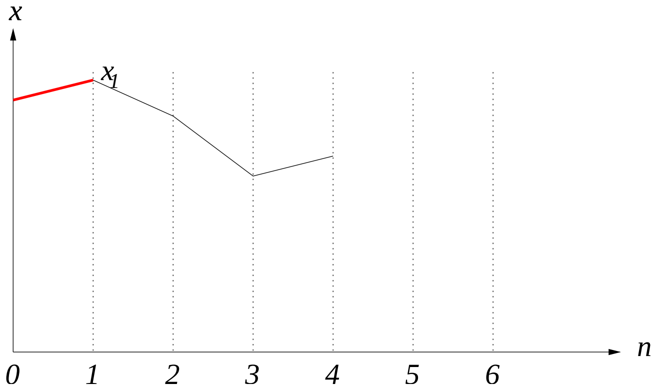

black = predictions (open loop optimization)
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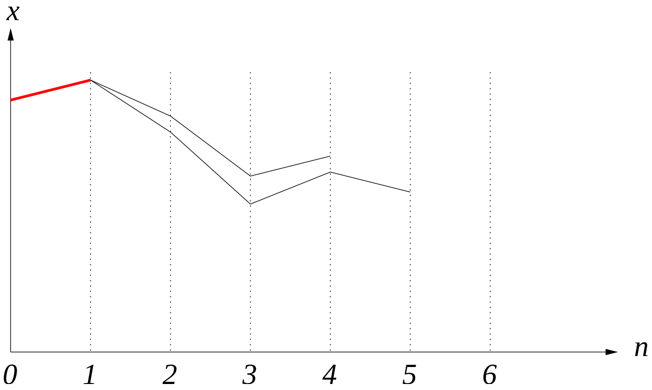

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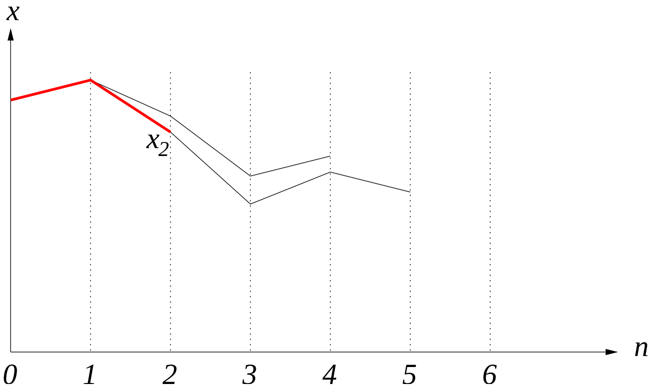

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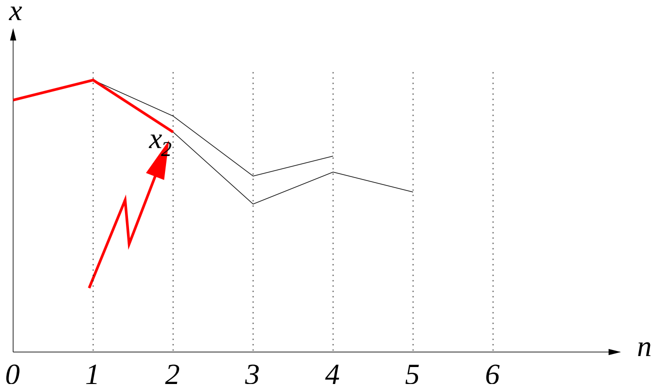

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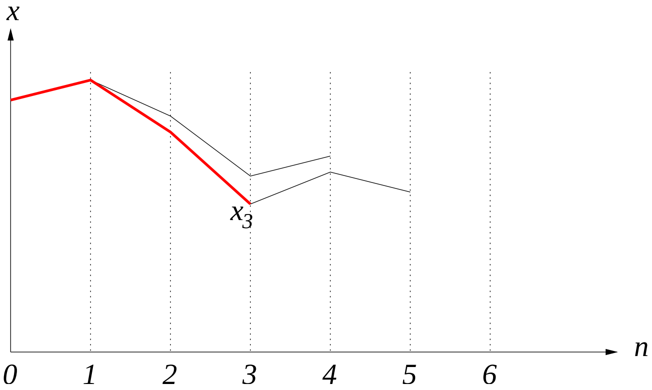

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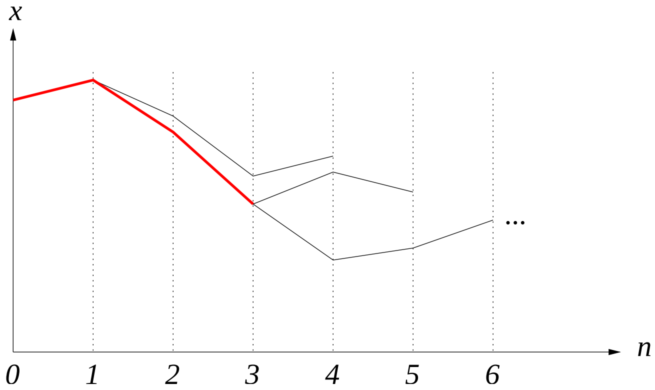

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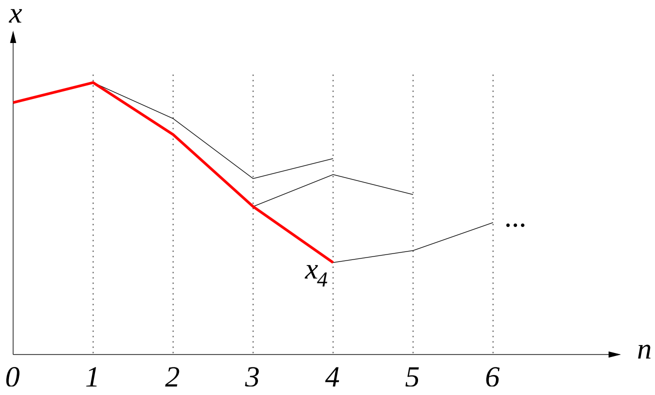

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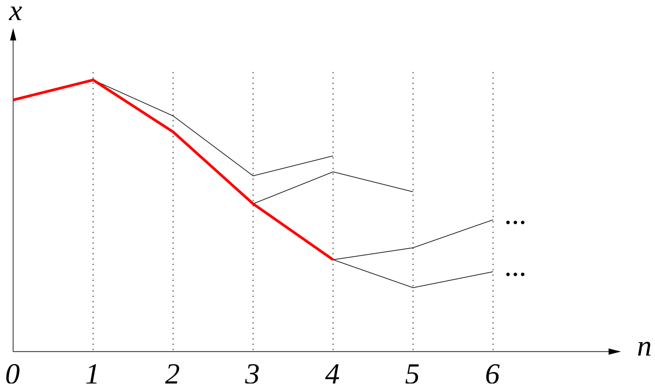

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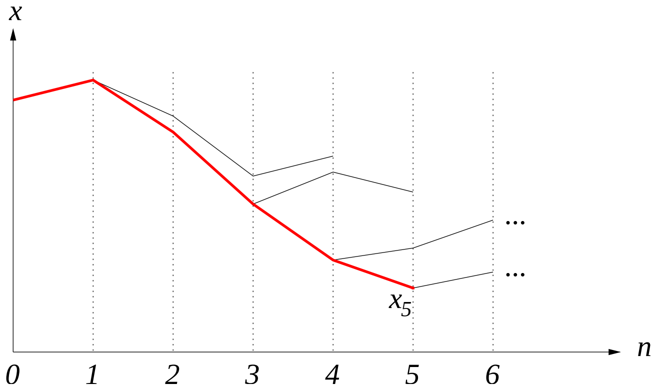

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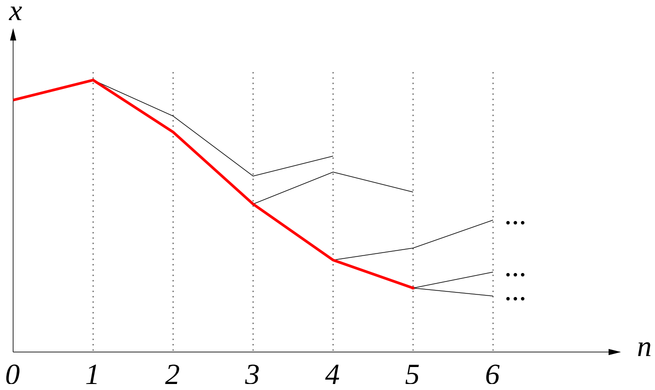

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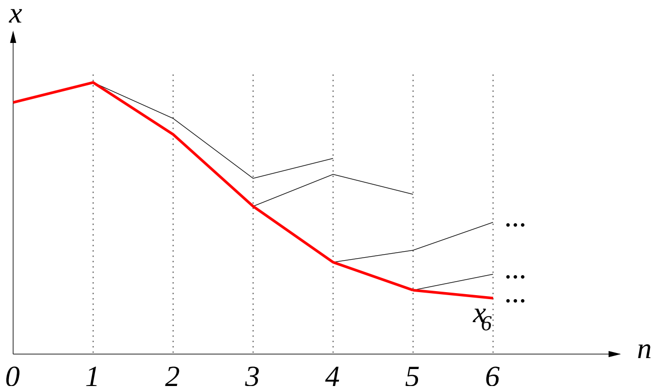

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Rigorous formulation

Denote (successful) transmission times by n_i, $i = 1, 2, \ldots$

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$$F_N(x(n_i), k) = u^*(k), \quad k = 0, 1, 2, \ldots, M - 1$$

and implements

$$F_N(x_{n_i}, 0),\ F_N(x_{n_i}, 1),\ \ldots,\ F_N(x_{n_i}, m_i - 1)$$

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Note m_i is unknown at time n_i

Stability theorem

Theorem: If there exists $\alpha \in (0,1]$ such that the relaxed dynamic programming inequality

$$V_N(x(m,x_0,u^*)) \le V_N(x) - \alpha \sum_{k=0}^{m-1} \ell(x(m,x_0,u^*),u^*(m))$$

holds for all $m = 1,\ldots,M$, then asymptotic stability follows for the MPC closed loop with arbitrary transmission times n_i, $i \in \mathbb{N}$, satisfying $n_{i+1} - n_i \ge M$.

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Furthermore, V_N is Lyapunov function at the transmission times n_i and we get the suboptimality estimate

$$J_\infty(x,F_N) \le V_\infty(x)/\alpha$$

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Computation of α

Again, for C, σ-exponentially controllable systems, this $\alpha = \alpha(C, \sigma, N, m)$ can be explicitly computed:

$$\alpha = 1 - \frac{\prod_{i=m+1}^{N} (\gamma_i - 1) \prod_{i=N-m+1}^{N} (\gamma_i - 1)}{\left(\prod_{i=m+1}^{N} \gamma_i - \prod_{i=m+1}^{N} (\gamma_i - 1)\right)\left(\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1)\right)}$$

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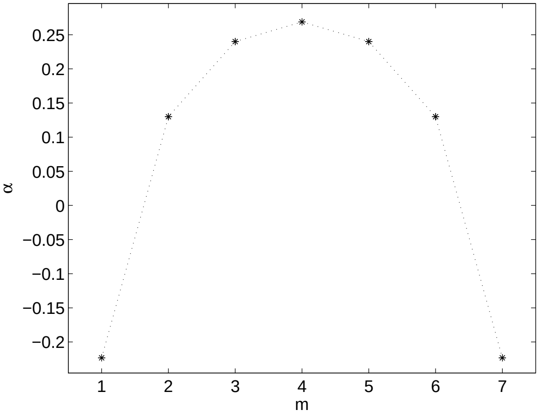
Note: at first, this yields a stability criterion for each fixed control horizon m only. However, since we get a common Lyapunov function V_N, stability carries over to varying control horizon m_i

Example



$\alpha(C, \sigma, N, m)$ for $C = 2$, $\sigma = 0.68$, $N = 8$, $m = 1, \ldots, 7$

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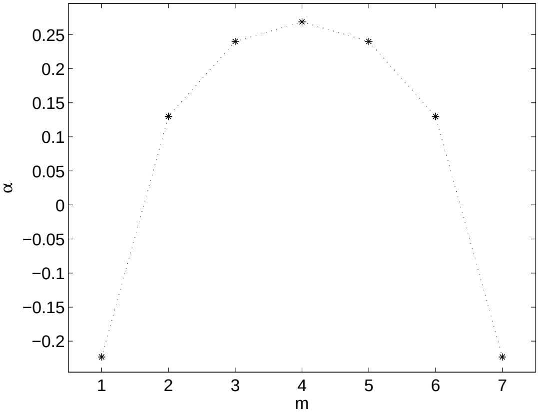

$\alpha(C,\sigma,N,m)$ for $C=2$, $\sigma=0.68$, $N=8$, $m=1,\ldots,7$

This symmetry and monotonicity is not a coincidence

Properties of α

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$$\alpha(N,m) = \alpha(N,N-m), \;\; m = 1,\ldots,N-1$$

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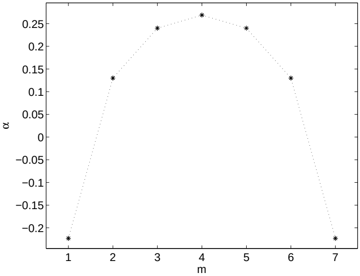
Corollary: If N is such that all C, σ-exponentially controllable systems are stabilized with "classical" MPC ($m = 1$), then they are stabilized for arbitrary varying control horizons $m_i \in \{1, \ldots, N-1\}$

Monotonicity in simulation examples



The monotonicity means that when enlarging the control horizon the closed loop performance first improves and then becomes worse, again

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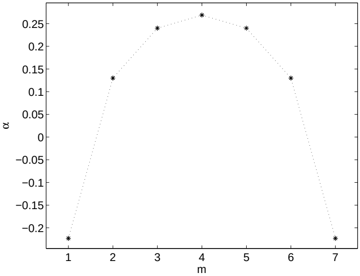


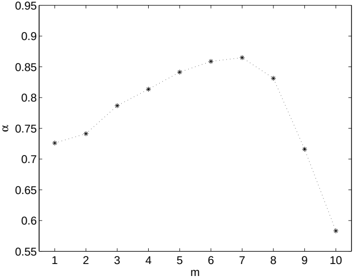
The monotonicity means that when enlarging the control horizon the closed loop performance first improves and then becomes worse, again

While we cannot exactly recover the symmetry in simulation examples, we can observe this monotonicity

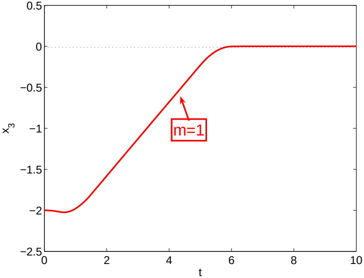
Example: linearized inverted pendulum

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} u, \qquad x_0 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

sampling time $T = 0.5$, $\ell(x,u) = 2\|x\|_1 + 4\|u\|_1$, $N = 11$



α after 1 MPC step



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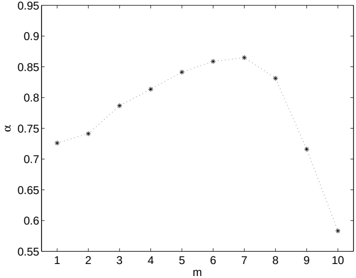

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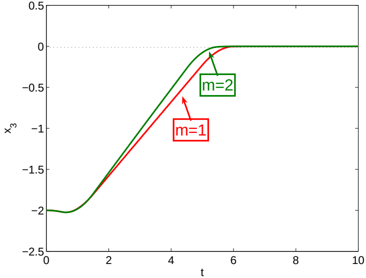

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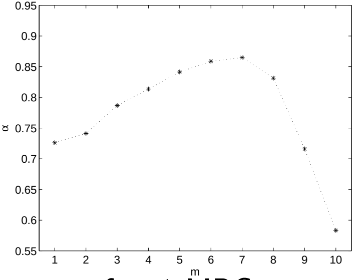

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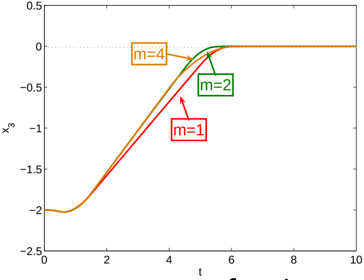

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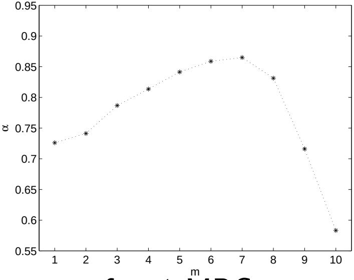

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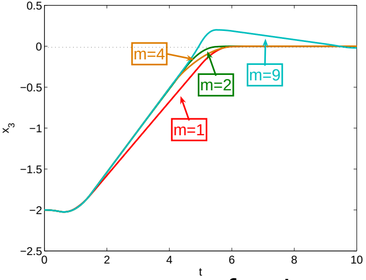

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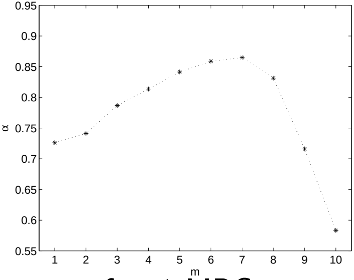

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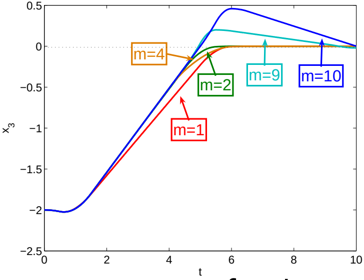

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Relaxed dynamic programming in distributed MPC — some ideas

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- some communication between the subsystems is necessary, often restricted to communication with neighbors

Structural properties, e.g., whether the subsystems are physically coupled or only coupled via the optimization objective play an important role for the design of the scheme

Communication during optimization

If we assume sufficiently fast communication, then subsystems may exchange information during the iterative optimization in each time step

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Ideas: Use a relaxed dynamic programming type condition

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Design the optimzation such that this condition is satisfied after few iterations.

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Idea: Replace the constraints by suitable "decentralized" controllability conditions under which stability and — if possible — suboptimality can be shown with our techniques

Asynchronous Optimization

So far, we implicitly assumed that optimization in the the subsystems is performed synchronously

However, for various reasons it may be preferable that different subsystems optimize at different time instances

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Idea: use similar techniques as in the networked setting in order to analyze stability and suboptimality

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