Two Problems on Minimality in RLC Circuit Synthesis

Timothy H. Hughes[1], Jason Z. Jiang[2], and Malcolm C. Smith[1]

Introduction

A (one-port) RLC circuit is an interconnection of resistors and reactive elements (inductors and capacitors) with a pair of driving-point terminals. The circuit may be characterised by its *impedance*, which is the transfer function from an applied current to the induced voltage across the driving-point terminals. A given circuit has a unique impedance which may be routinely calculated from information about the constituent elements and their interconnections. It is well known that this impedance is necessarily *positive-real* [2], and that the *McMillan degree* of the impedance is less than or equal to the number of reactive elements in the circuit [9]. RLC circuit synthesis is concerned with obtaining RLC circuit *realisations* for a given impedance. Several key contributions to the synthesis problem were made in the first half of the twentieth century [1, 2, 5], but work on the subject petered out in the early 1970s. There has been a recent resurgence of interest in the field due to the introduction of a new mechanical component—the *inerter* [21]—which permits an exact analogy between electrical and mechanical circuits. The application of the inerter to mechanical vibration problems (e.g. vehicle suspensions) motivates a fresh look at questions of *minimality* and *efficiency of realisation* that have remained unresolved in the field. In this article, we discuss two such open problems.

Problem 1: *In order to synthesise the entire class of impedances realisable with an RLC circuit containing n reactive elements, what is the least number of resistors required?*

In [15], Lin shows that the impedance of any series-parallel network consisting of one inductor, one capacitor, and any number of resistors, can be realised by a network consisting of no more than one capacitor, one inductor and three resistors. The paper concludes with the question: 'will the theorem stated in this correspondence remain valid if the restriction "series-parallel" is removed?' Despite Foster's claim in [6] that no more than three resistors are necessary, no proof was available until Reichert's German language publication [17]. A crucial topological argument in Reichert's proof has been expanded, and a complete reworking of the proof has been published in [13].

A systematic survey of RLC circuits containing two reactive elements appeared in the Master's thesis of Ladenheim [14]. There, all networks with at most three resistors and at most two reactive elements are considered and reduced to 108 networks by various transformations. Realisability conditions on parameters of the corresponding biquadratic impedance are listed for each network. However, the totality of biquadratics which may be realised by this set of networks is not obvious from the result. A restudy of the class of networks in [14] has been carried out in [12]. A complete classification of the realisability of biquadratics by means of five-element networks with two reactive elements has been presented. The approach is facilitated by the introduction of the concept of regular positive-real functions. It has been shown that a biquadratic can be realised by a series-parallel network with two reactive elements if and only it is regular. Furthermore, six such series-parallel networks, each with three resistors, can realise all regular biquadratics. It has also been shown that the only five element networks capable of realising non-regular biquadratics are a pair of bridge networks studied by Foster and Ladenheim [7]. Based on the regularity concept, a new proof of Lin's result on minimum reactive

[1]Timothy H. Hughes and Malcolm C. Smith are with the Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, U.K., thh22@cam.ac.uk, mcs@eng.cam.ac.uk.

[2]Jason Z. Jiang is with the Department of Engineering, University of Bristol, Bristol, BS8 1TR, U.K., z.jiang@bristol.ac.uk.

series-parallel synthesis of the biquadratic has also been obtained [11].

The present problem is therefore solved in the case $n = 2$. Namely, among RLC realisations, three resistors suffice to realise the entire class of impedances realisable with two reactive elements. Moreover, six series-parallel networks and two bridge networks can realise the whole class. In the remainder of this section, we discuss three significant challenges to the extension of these results to $n > 2$.

Both Lin's approach and the proof based on regularity rely on an analysis of all series-parallel network structures with two reactive elements. As the number of reactive elements is increased, the number of candidate network structures grows very rapidly [19,22]. This makes the enumeration and analysis process more challenging.

A second difficulty is the complexity of algebraic manipulations for impedance functions with McMillan degree higher than two. Both Reichert's approach and the identification of conditions for regularity require an understanding of the algebraic constraints on those biquadratics that can be realised by networks with two reactive elements and three resistors. These constraints are polynomial inequalities in the coefficients of the biquadratic function. A canonical form for the biquadratic was introduced in [17], and graphical illustrations were used to facilitate the identification of these inequalities. The number of coefficients in the impedance function increases linearly with the number of reactive elements. Consequently, it will be much harder to find the algebraic conditions representing the totality of impedances that are realisable with n reactive elements for the cases $n > 2$.

A third challenge is the absence of necessary and sufficient realisability conditions for l-port resistive networks with $l > 3$. See "The classical n-port resistive synthesis problem" by M.Z.Q. Chen in the present problem collection for a further discussion. A key step of Reichert's proof relies on Cederbaum's *paramountcy* conditions for the realisation of a resistive 3-port. However, it has been shown that paramountcy is a necessary but not a sufficient realisability condition for resistive networks with more than three ports. Indeed, a counter-example for $l = 4$ was given in [3,23].

In summary, this problem is solved in the case $n = 2$, but there exist significant challenges in the cases $n > 2$.

Problem 2: *Among RLC realisations, what is the minimum number of reactive elements required to realise all positive-real functions of McMillan degree n?*

In [2] it was shown that the impedance of an RLC circuit is necessarily positive-real. The paper [1] by Bott and Duffin presented the first constructive procedure for obtaining a circuit realisation for any given positive-real function. A slight improvement on the Bott-Duffin procedure was published in the three papers [4,16,18]. The number of reactive elements required by both the Bott-Duffin procedure and its improvement grows exponentially with the McMillan degree of the function being realised. In contrast, there are many positive-real functions of a given McMillan degree n that are realised by RLC circuits containing exactly n reactive elements. This has led to speculation about whether these procedures can be improved.

The original procedure of Bott and Duffin results in a *series-parallel* network. The procedure is inductive, and uses a preliminary step known as the *Foster preamble*. The Foster preamble either provides a complete realisation for a given positive-real function, or provides a partial realisation and terminates with a special type of positive-real function known as a *minimum function*. The key contribution in [1] is the proof that any minimum function $H(s)$ can be realised by a series-parallel network comprised of six reactive elements and two subnetworks whose impedances have McMillan degree at least two fewer than that of $H(s)$. In the case of a *biquadratic* minimum function (one whose McMillan degree is two), the procedure requires six reactive elements and two resistors. In a recent paper [10] it is shown that six is the least possible number of reactive elements among all series-parallel realisations of biquadratic minimum functions.

The papers [4, 16, 18] show that it is possible to realise any given minimum function $H(s)$ with an RLC circuit comprised of five reactive elements and two subnetworks whose impedances have McMillan degree at least two fewer than that of $H(s)$. The circuits obtained are not series-parallel. In [8] it is shown that five is the least possible number of reactive elements among all RLC circuits for the realisation of almost all biquadratic minimum functions.

The present problem is thus solved in the case $n = 2$. Namely, among RLC realisations, five reactive elements are required to realise all positive-real functions of McMillan degree two. Moreover, among *series-parallel* networks, six reactive elements are required. These are precisely the numbers of reactive elements used in the procedures in [4, 16, 18] and [1] respectively. In the remainder of this section, we discuss three significant challenges to solving this problem in the case $n > 2$.

In the case of biquadratic positive-real functions, it turns out that the minimum functions prove particularly difficult to realise. It is not clear whether this will remain the case for positive-real functions of higher McMillan degree. Indeed, it is shown in [10] that there are certain minimum functions of McMillan degrees three, four and five that are realised by a series-parallel network containing five reactive elements, which is considerably fewer than the number required by the Bott-Duffin procedure.

A key step in demonstrating minimality is to reduce the number of networks which require consideration from an infinite set to a finite set (since our focus is on the number of reactive elements, and no restriction is placed on the number of resistors which may be used). Even if a suitable finite set of networks is identified, this set may still be very large, considering the significant growth in the number of networks which contain at most m elements as m is increased [19, 22].

A third challenge is the complexity of algebraic manipulations. The impedance of a given circuit is a ratio of two polynomial functions in the Laplace domain variable and the element parameters. The task of determining whether a circuit realises a given positive-real function is then equivalent to that of determining whether there exist non-negative element parameters to satisfy a system of polynomial equations. This latter problem is commonly referred to as *quantifier elimination* [20, Chapter 14]. The complexity of such problems is known to increase considerably as the number of variables to eliminate, in this case corresponding to the number of elements in the circuit, is increased.

In conclusion, the present problem is solved in the case $n = 2$, but there are significant challenges in extending this to $n > 2$.

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