

On the asymptotic behavior of differentially positive systems

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Abstract

Monotonicity [5] - a property that requires trajectories to preserve some partial order relation - strongly restricts the asymptotic behavior of a dynamical system. In fact, (strict) monotonicity guarantees that almost every trajectory converges to a fixed point [3]. This property is the fundamental result of the theory of monotone systems. Differential positivity [2] is a recent generalization of monotonicity. Here, the partial order is allowed to vary from point to point, that is, it is a partial order with respect to a smooth but not necessarily constant cone field. We conjecture that *almost every trajectory of a (uniformly strictly) differentially positive system converges to a simple attractor*.

A glance into differential positivity

A linear system is positive if there is a (pointed, convex, solid) cone $\mathcal{K} \subseteq \mathbb{R}^n$ which is forward invariant for the system trajectories [1]. The trajectories of a linear positive system preserve the partial order given by \mathcal{K} [5]. The trajectories of a nonlinear system are locally ordered if the system is differentially positive, that is, if the linearized dynamics is positive [2]. Positivity must be intended in a generalized sense, since the linearized dynamics along the trajectory $x(\cdot)$ lives in the tangent space $T_{x(t)}\mathcal{X}$ of the system state manifold \mathcal{X} . The cone of linear positivity becomes a (smooth) cone field $\mathcal{K}(x) \in T_x\mathcal{X}$. Given the system $\dot{x} = f(x)$, the forward invariance property reads

$$\delta x(t_0) \in \mathcal{K}(x(t_0)) \quad \Rightarrow \quad \delta x(t) \in \mathcal{K}(x(t)) \quad \forall t \geq t_0, \quad (1)$$

where $(x(\cdot), \delta x(\cdot))$ is any trajectory of the prolonged system given by the pairing of $\dot{x} = f(x)$ and its linearization $\dot{\delta x} = \partial f(x)\delta x$. Differential positivity coincides with monotonicity when the state manifold is a vector space and the cone field is constant.

Uniformly strictly differentially positive systems satisfy (1) with the stronger condition that the boundaries of the cone at $x(t)$ are mapped into the interior of the cone at $x(t+T)$ uniformly, for some $T > 0$, as shown in Figure 1.

The projective contraction of linear positive systems [1] extends to uniformly strictly differentially positive systems [2, Section VI], leading to the characterization of the so-called *Perron-Frobenius vector field* $w(x) \in T_x\mathcal{X}$, a continuous vector field of unit length such that

$$\lim_{t \rightarrow \infty} \frac{\delta x(t)}{|\delta x(t)|} = w(x(t)) \quad (2)$$

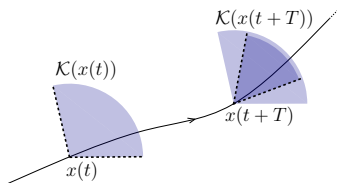


Figure 1: Contraction of the cone field along the system trajectories.

for any trajectory $(x(\cdot), \delta x(\cdot))$ of the prolonged system whose initial condition $\delta x(t_0) \in \mathcal{K}(x(t_0))$. The effect on the system behavior of (2) can be easily gathered by looking at trajectories $x(\cdot)$ whose vector field f at the initial condition $x(t_0)$ satisfies $f(x(t_0)) \in \mathcal{K}(x(t_0))$, $|f(x(t_0))| \neq 0$. Since $\frac{d}{dt}f(x(t)) = \partial f(x(t))f(x(t))$, we have that $\lim_{t \rightarrow \infty} \frac{f(x(t))}{|f(x(t))|} = w(x(t))$. Thus, asymptotically, the trajectories of the system merge to a *Perron-Frobenius curve*, i.e. an integral curves of the Perron-Frobenius vector field.

Simple and complex behaviors

An appealing property of differential positivity is a characterization of limit sets that is reminiscent of Poincaré-Bendixon theorem for planar systems: solutions in a compact invariant set that does not contain a fixed point must converge to a unique limit cycle.

The essence of that property is a fairly simple geometric argument that shows that any bounded open forward invariant region $\mathcal{R} \subseteq \mathcal{X}$ for which the linearized dynamics remains *bounded* contains only simple attractors (fix points, limit cycles, a set of fixed points and connecting arcs compatible with). In fact, the boundedness of $\delta x(\cdot)$ and the projective contraction (2) guarantee that any recurrent set in the state space contracts along the directions transversal to the Perron-Frobenius vector field.

The situation is more involved for trajectories along which the linearized dynamics are not bounded. In this case, the Perron-Frobenius vector field $w(x(t))$ characterizes the direction of strongest sensitivity, that is, the direction of maximal divergence between $x(\cdot)$ and the neighboring trajectories. For example, the Perron-Frobenius vector field is everywhere transversal to an unstable limit cycle and it is tangent to the unstable manifold of a saddle point.

The local information of sensitivity combined with the global information on the shape of the Perron-Frobenius curves opens the way to the study of basins of attraction. In Figure 2.I, those curves define a very regular foliation of the attractor \mathcal{A} (explored by the unstable trajectory $x(\cdot)$). In such a case, any trajectory $z(\cdot)$ from an initial condition $z(t_0)$ in the (infinitesimal) neighborhood of \mathcal{A} , $z(t_0) \notin \mathcal{A}$, is forced to move away from \mathcal{A} , in accordance with the unstable direction defined by the Perron-Frobenius vector field, and cannot come back. As a consequence, the basin of attraction of \mathcal{A} is a set of dimension at most $n - 1$, where n is the dimension of the state manifold. If \mathcal{A} is a chaotic attractor, for example, then from almost every initial condition the trajectories of the system do not converge to the chaotic attractor.

In general, however, it is difficult to characterize the shape of the Perron-Frobenius curves. Potentially, they may puncture the attractor at a point, fold, and come back to the attractor through another point, even without puncturing the attractor (in this last case, the strongest sensitivity direction is tangent to the attractor). Thus, in general, ruling out attractive chaotic

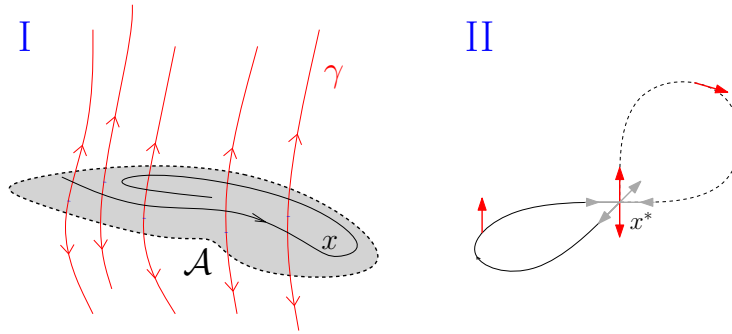


Figure 2: [I] The integral curves of the Perron-Frobenius vector field in red define a foliation of the attractor \mathcal{A} . [II] A saddle with two homoclinic orbits. The dashed homoclinic orbit is not compatible with the local order at the saddle point.

behaviors is an open question. Still, neighboring trajectories must preserve the local order given by the cone field, which provides a clue on the correctness of the conjecture.

The issue can be illustrated through the analysis of homoclinic orbits, which are at the core of Smale's construction of hyperbolic strange attractors [4, pp. 843-852]. The stable and unstable manifolds of the saddle x^* in Figure 2.II have dimension 1 and 2, respectively. Because of the orientation of the Perron-Frobenius vector field at the saddle point (red arrow), the homoclinic orbit $x(\cdot)$ on the right (dashed) satisfies $\frac{f(x(t))}{|f(x(t))|} = w(x(t))$ for all t . However, by construction $\lim_{t \rightarrow \infty} \frac{f(x(t))}{|f(x(t))|} \neq w(x^*)$, contradicting (2). Indeed, the dashed homoclinic orbit is ruled out by differential positivity. The homoclinic orbit on the left part of the figure (solid) is instead compatible with differential positivity. In such a case, however, the Perron-Frobenius vector field is necessarily nowhere tangent to the curve. Indeed, the solid homoclinic orbit defines a repulsive set for any neighboring solution. The analysis of the homoclinic orbits suggests that aperiodic solutions of a strange attractors are compatible with uniform strict differential positivity only if the Perron-Frobenius vector field is everywhere transversal to the attractor.

References

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